Parabolic PDEs on Evolving Spaces

by

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Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Mathematics Institute

February 2016

THE UNIVERSITY OF WARWICK

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Acknowledgements

My foremost thanks to my supervisors, Charlie and Björn, who introduced me to many interesting areas in PDEs and provided me with many opportunities in the academic world. It was a pleasure to work with them.

I must thank Hans for an enjoyable collaboration and for his advice about being a mathematician. I thank James for his help, and also for his course on PDEs which initiated my interest in the topic. I am also grateful to those involved in the creation and administration of MASDOC, and to EPSRC for funding me.

My grateful thanks to my thesis examiners, James and Juan Luis, for many excellent suggestions which helped to improve the thesis and for their general advice.

I would like to thank my departmental friends and colleagues (past and present); in particular Alex, Maha, and Simon for the numerous coffee trips and also Owen, Michael, and Andrew.

Outside Warwick, special thanks to my friends Matt, Jeff, Paul, Dave, and Ross.

Finally, I gratefully acknowledge the support of my parents and family, to whom I dedicate this thesis.

Declarations

This thesis is based on four papers that I co-wrote with one or both of my supervisors.

- Large parts of the introduction is taken from the abstracts of [1, 2, 3, 4]
- Chapter 1 is taken from [1] except for §1.2.3, which is taken from [3]
- Chapter 2 is taken from [2]
- Chapter 3 is taken from [3]
- Chapter 4 is a much-expanded version of [4], which has been accepted for publication.

Portions of the first three chapters of this work were undertaken when I participated at the Isaac Newton Institute in Cambridge during the *Free Boundary Problems and Related Topics* programme.

I declare that to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated. This thesis has not been submitted for a degree at any other university.

Abstract

This thesis is concerned with the well-posedness of solutions to certain linear and nonlinear parabolic PDEs on evolving spaces. We first present an abstract framework for the formulation and well-posedness of linear parabolic PDEs on abstract evolving Hilbert spaces. We introduce new function spaces and a notion of a weak time derivative called the weak material derivative for this purpose. We apply this general theory to moving hypersurfaces and Sobolev spaces and study four different linear problems including a coupled bulk-surface system and a dynamical boundary problem. Then we formulate a Stefan problem itself on an evolving surface and consider weak solutions given integrable data through the enthalpy approach, using a generalisation to the Banach space setting of the function spaces introduced in the abstract framework. We finish by studying a nonlocal problem: a porous medium equation with a fractional diffusion posed on an evolving surface and we prove well-posedness for bounded initial data.

Introduction

The overarching theme of this thesis is the existence, uniqueness and continuous dependence of solutions to parabolic equations on evolving spaces. We will study linear equations on abstract evolving Hilbert spaces and on moving hypersurfaces and domains, and also some nonlinear problems on moving hypersurfaces.

We present in Chapter 1 an abstract framework for treating the theory of well-posedness of solutions to abstract linear parabolic PDEs on evolving Hilbert spaces. These equations have the form

$$\dot{u}(t) + A(t)u(t) = f(t)$$

where the equality is in $V^*(t)$, with $V(t)$ a Hilbert space for each $t \in [0, T]$, and $A(t)\colon V(t) \to V^*(t)$ is a linear elliptic operator. This theory is applicable to variational formulations of PDEs on evolving spatial domains including moving hypersurfaces. We formulate an appropriate time derivative on evolving spaces called the material derivative and define a weak material derivative in analogy with the usual time derivative in fixed domain problems; our setting is abstract and not restricted to evolving domains or surfaces. Then we show well-posedness of a certain class of linear parabolic PDEs under some assumptions on the parabolic operator and the data.

Next, in Chapter 2, we consider existence and uniqueness for several examples of linear parabolic equations formulated on moving hypersurfaces. Specifically, we study in turn a surface heat equation, an equation posed on a bulk domain, a novel coupled bulk-surface system:

$$\begin{aligned}
\dot{u}(t) - \Delta_\Omega u(t) + u(t)\nabla_\Omega \cdot \mathbf{w}(t) &= f(t) && \text{on } \Omega(t)\\
\dot{v}(t) - \Delta_\Gamma v(t) + v(t)\nabla_\Gamma \cdot \mathbf{w}(t) + \nabla_\Omega u(t) \cdot \nu(t) &= g(t) && \text{on } \Gamma(t)\\
\nabla_\Omega u(t) \cdot \nu(t) &= \beta v(t) - \alpha u(t) && \text{on } \Gamma(t)\\
u(0) &= u_0 && \text{on } \Omega(0)\\
v(0) &= v_0 && \text{on } \Gamma(0),
\end{aligned}$$

and an equation with a dynamic boundary condition:

$$\begin{aligned}
\Delta v(t) &= 0 && \text{on } \Omega(t)\\
\dot{u}(t) + \frac{\partial v(t)}{\partial \nu(t)} + u(t) &= f(t) && \text{on } \Gamma(t)\\
u(0) &= v_0 && \text{on } \Gamma(0).
\end{aligned}$$

Above, $\nu(t)$ denotes the unit normal to the bounded domain $\Omega(t)$ and $\Gamma(t) := \partial\Omega(t)$. In order to prove the well-posedness, we make use of the abstract framework presented in Chapter 1; we first show that it can applied to the case of evolving hypersurfaces, and then we demonstrate the utility of the framework to the aforementioned problems.

In Chapter 3, we formulate a Stefan problem on an evolving hypersurface and study the well-posedness of weak solutions given L^1 data. The resulting nonlinear equation, posed on a moving compact hypersurface $\Omega(t) \subset \mathbb{R}^{n+1}$, is

$$\begin{aligned}
\dot{e}(t) - \Delta_{\Omega(t)} u(t) + e(t)\nabla_{\Omega(t)} \cdot \mathbf{w}(t) &= f(t) && \text{on } \Omega(t)\\
e(0) &= e_0 && \text{on } \Omega(0)\\
e &\in \mathcal{E}(u),
\end{aligned}$$

where the energy (or enthalpy) $\mathcal{E}\colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is defined by $\mathcal{E}(r) = r\chi_{\{r<0\}} + [0,1]\chi_{\{r=0\}} + (r+1)\chi_{\{r>0\}}$. To prove the well-posedness, we first develop function spaces and results to handle equations on evolving surfaces in order to give a natural treatment of the problem. Then we consider the existence of solutions for L^∞ data; this is done by regularisation of the nonlinearity. The regularised problem is solved by a fixed point theorem and then uniform estimates are obtained in order to pass to the limit. By using a duality method we show continuous dependence which allows us to extend the results to L^1 data.

Finally, in Chapter 4, we consider the existence, uniqueness, and the L^1-

contractivity of weak solutions to the fractional porous medium equation

$$\begin{aligned} \dot{u}(t) + (-\Delta_{\Gamma(t)})^{1/2}(u^m(t)) + u(t)\nabla_{\Gamma(t)} \cdot \mathbf{w}(t) = 0 \quad &\text{on } \Gamma(t) \\ u(0) = u_0 \quad &\text{on } \Gamma(0) \end{aligned}$$

on an evolving surface $\Gamma(t)$, for $m \geq 1$. We reformulate the equation as a local problem on the semi-infinite cylinder $\Gamma(t) \times [0, \infty)$, regularise the porous medium nonlinearity and truncate the cylinder. Then we pass to the limit first in the truncation parameter and then in the nonlinearity. The identification of limits is done using the theory of subdifferentials of convex functionals. In order to facilitate all of this, we begin by studying (in the setting of closed Riemannian manifolds and Sobolev spaces) the fractional Laplace–Beltrami operator which can be seen as the Dirichlet-to-Neumann map of a harmonic extension problem. A truncated harmonic extension problem will also be examined and convergence results of the solution to the (untruncated) harmonic extension will be given (these results are used in passing to the limit in the truncation described above). This theory is of course independent of the fractional porous medium equation and will be of use generally in the study of fractional elliptic and parabolic problems on manifolds. We will also consider some related extension problems on evolving hypersurfaces which will provide us with the language to formulate and solve the fractional porous medium equation (amongst others) on evolving hypersurfaces.

The thesis will be concluded with some ideas for further work.

Before we proceed, let us state the following compactness result which we will refer to throughout this work simply as Aubin–Lions for convenience. We make use of the notation $\hookrightarrow$ and $\overset{c}{\hookrightarrow}$ to denote (respectively) a continuous embedding and a compact embedding.

Theorem 0.0.1 (Aubin–Lions–Simon, Theorem II.5.16 in [24])**.** Let $B_0 \overset{c}{\hookrightarrow} B_1 \hookrightarrow B_2$ be Banach spaces and let p, q be such that $1 \leq p, q \leq \infty$. Define

$$W = \{u \in L^p(0, T; B_0) \mid u' \in L^q(0, T; B_2)\}.$$

1. If $p < \infty$, then $W \overset{c}{\hookrightarrow} L^p(0, T; B_1)$.
2. If $p = \infty$ and $q > 1$, then $W_{p,q} \overset{c}{\hookrightarrow} C^0([0, T]; B_1)$.

Chapter 1

An abstract framework for parabolic PDEs on evolving spaces

1.1 Introduction

Partial differential equations on evolving or moving domains are an active area of research [38], [50], [98], [99], partly because their study leads to interesting analysis but also because models describing applications such as biological and physical phenomena can be better formulated on evolving domains (including hypersurfaces) rather than on stationary domains. For example, see [9], [66] for studies of pattern formation on evolving surfaces, [68] for the modelling of surfactants in two-phase flows, [52] for the modelling and numerical simulation of dealloying by surface dissolution of a binary alloy (involving a forced mean curavture flow coupled to a Cahn–Hilliard equation), [57] (and the references therein for applications) for the analysis of a diffuse interface model for a linear surface PDE, and [58] for the modelling and simulation of cell motility.

One aspect to consider in the study of such equations is how to formulate the space of functions that have domains which evolve in time. Taking a disjoint union of the domains in time to form a non-cylindrical set is standard: see [17], [128], [99] for example. Of particular interest is [79] where the problem of a semilinear heat equation on a time-varying domain is considered; the set-up of the evolution of the domains is comparable to ours and similar function space results are shown (in the setting of Sobolev spaces). In [16], the authors define Bochner-type spaces by considering a continuous distribution of domains $\{\Gamma(t)\}_{t\in[0,T]} \subset \mathbb{R}^n$ that are embedded in a larger domain Γ. The aim of our work is to accommodate not only evolving domains but arbitrary evolving spaces. Our method, which follows that of [125], is somewhat different to the aforementioned and contains an attachment to standard Bochner spaces in a fundamental way.

A common procedure for showing well-posedness of equations on evolving domains involves a transformation of the PDE onto a fixed reference domain to which abstract techniques from functional analysis are applied [87], [100], [5], [125]. For example, in [125], the heat equation

$$\dot{u}(t) - \Delta_{\Gamma(t)} u(t) + u(t) \nabla_{\Gamma(t)} \cdot \mathbf{w}(t) = f(t) \qquad \text{in } H^{-1}(\Gamma(t)) \tag{1.1}$$

on an evolving surface $\{\Gamma(t)\}_{t \in [0,T]}$ is considered, with $\mathbf{w}$ representing the velocity field. The equation is pulled back onto a reference domain $\Gamma(s)$ and standard results on linear parabolic PDEs are applied. A Faedo–Galerkin method (see [11] for a historical overview of the method) is used in [100] (for a different PDE), where the evolving domain is represented by perturbations of the reference domain and *a priori* estimates are derived for a linearised problem. An adapted Galerkin method that uses the pushforward of eigenfunctions of the Laplace–Beltrami operator on $\Gamma(0)$ to form a countable dense subset of $H^1(\Gamma(t))$ is employed in [48] for the advection-diffusion equation (1.1). We abstract this approach for one of our results. Well-posedness for the same class of equations is obtained in [98] by employing a variational formulation on space-time surfaces and utilising a standard generalisation of the classical Lax–Milgram theorem used by Lions for parabolic equations. We also employ this Lions–Lax–Milgram approach in our abstract setting.

As we have seen, there is much literature in which certain equations on evolving domains are studied, however, to the best of our knowledge, there is no unifying theory or framework that treats parabolic PDEs on *abstract* evolving spaces. The main aim of this chapter is to provide this abstract framework. More specifically, given a linear time-dependent operator $A(t)$ we study well-posedness of parabolic problems of the form

$$\dot{u}(t) + A(t)u(t) = f(t) \tag{1.2}$$

as an equality in $V^*(t)$, with $V(t) \subset H(t)$ a Hilbert space for each $t \in [0, T]$. A main feature of our work is the definition of an appropriate time derivative on evolving spaces *in an abstract setting*. When the said spaces are simply L^p spaces on curved or flat surfaces in $\mathbb{R}^n$ that evolve in time, it is commonplace to take the material derivative

$$\dot{u}(t) = u_t(t) + \nabla u(t) \cdot \mathbf{w}(t)$$

from continuum mechanics as the natural time derivative (here $\mathbf{w}$ is again the velocity field of the moving domain or surface). But when we have arbitrary spaces that may have no relationship whatsoever with $\mathbb{R}^n$ it is not at all clear what the $\dot{u}(t)$ in (1.2) should mean. We will deal with this issue and define a material derivative and a weak material derivative for the abstract case. Our framework relies on the existence of a family of (pushforward) maps ϕ_t for $t \in [0, T]$ that allow us to map the initial spaces $V(0)$ and $H(0)$ to the spaces $V(t)$ and $H(t)$. A particular realisation of these maps ϕ_t in the case of, for example, the heat equation (1.1) takes into account the evolution of the surfaces $\Gamma(t)$ and hence ϕ_t will be a flow map defined by the velocity field $\mathbf{w}$. This family of parametrisations ϕ_t is not unique; see Remark 1.2.9. Although one motivation behind this work is the analysis of equations on moving domains and hypersurfaces, the framework can also be useful for problems on fixed domains where, for example, $H(t)$ and $V(t)$ may be weighted Lebesgue–Sobolev spaces with time-dependent weights.

Our belief is that the abstract procedure presented in this work is a clean and elegant approach to problems on moving domains. It is beneficial to have this abstract theory when working on complicated problems since the framework clearly indicates which results and assumptions need to be checked in order for there to be well-defined function spaces and properties relevant to the problem at hand. In addition, the theory and concepts presented here can be used as a foundation in extensions such as generalisations to the Banach space setting and the study of nonlinear problems. We also anticipate that our framework will benefit those working in numerical analysis since curved, flat, and evolving surfaces can all be treated with the same abstract procedure.

In Chapter 2, we will demonstrate the applicability of this abstract framework to the case of moving or evolving hypersurfaces. Four different examples of parabolic equations on moving hypersurfaces will be studied, and the well-posedness will be proved using the results we shall give in this chapter.

1.1.1 Outline

In §1.2, we start by setting up the function spaces and definitions required for the analysis and indeed the *statement* of equations of the form (1.2). We state our assumptions on the evolution of the spaces and define abstract strong and weak material derivatives (in analogy with the usual derivative and weak derivative utilised in fixed domain problems).

In §1.3 we precisely formulate the problem (1.2) that we consider and list the assumptions we make on A. Statements of the main theorems of existence, uniqueness, and regularity of solutions are given. The proof of one of these theorems is presented in §1.5. There, we make use of the generalised Lax–Milgram theorem. In §1.6 we formulate an adapted abstract Galerkin method similar to one described in [48] and use it to prove a regularity result.

1.1.2 Notation and conventions

Here and below we fix $T \in (0, \infty)$. When we write expressions such as $\phi_{(\cdot)}u(\cdot)$, our intention usually (but not always) is that both of the dots $(\cdot)$ denote the same argument; for example, $\phi_{(\cdot)}u(\cdot)$ will come to mean the map $t \mapsto \phi_t u(t)$. The notation X^* will denote the dual space of a Hilbert space X and X^* will be equipped with the usual induced norm $\|f\|_{X^*} = \sup_{x \in X \setminus \{0\}} \langle f, x \rangle_{X^*,X} / \|x\|_X$. We may reuse the same constants in calculations multiple times if their exact value is not relevant. Integrals will usually be written as $\int_S f(s)$ instead of $\int_S f(s)\, \mathrm{d}s$ unless to avoid ambiguity. Finally, we shall make use of standard notation for Bochner spaces; for example, see [63, §5.9].

1.2 Function spaces and functional analysis

As we mentioned above, in order to properly understand and express the equation (1.2), we need to devise appropriate spaces of functions. First, we begin with recalling some standard results regarding Sobolev–Bochner spaces from parabolic theory for the reader's convenience; a good reference for this is [41, §XVIII].

1.2.1 Standard Sobolev–Bochner space theory

Let $\mathcal{V}$ and $\mathcal{H}$ be Hilbert spaces and let $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$ be a Gelfand triple (i.e., all embeddings are continuous and dense and $\mathcal{H}$ is identified with its dual via the Riesz representation theorem). Recall that $u \in L^2(0, T; \mathcal{V})$ is said to have a *weak derivative* $u' \in L^2(0, T; \mathcal{V}^*)$ if there exists $w \in L^2(0, T; \mathcal{V}^*)$ such that

$$\int_0^T \zeta'(t)(u(t), v)_{\mathcal{H}} = -\int_0^T \zeta(t)\langle w(t), v\rangle_{\mathcal{V}^*,\mathcal{V}} \qquad \text{for all } \zeta \in \mathcal{D}(0,T) \text{ and } v \in \mathcal{V}, \tag{1.3}$$

and one writes $w = u'$. By $\mathcal{D}(0,T)$ we refer to the space of infinitely differentiable functions with compact support in $(0,T)$. We shall also make use of $\mathcal{D}([0,T];\mathcal{V})$; this is the space of $\mathcal{V}$-valued infinitely differentiable functions compactly supported in the *closed* interval $[0,T]$. A helpful characterisation of this space, from Lemma 25.1 in [127, §IV.25], is that $\mathcal{D}([0,T];\mathcal{V})$ is the restriction $\mathcal{D}((-\infty,\infty);\mathcal{V})|_{[0,T]}$ (the restriction to $[0,T]$ of infinitely differentiable $\mathcal{V}$-valued functions with compact support).

Lemma 1.2.1. The space

$$\mathcal{W}(\mathcal{V},\mathcal{V}^*) = \{u \in L^2(0,T;\mathcal{V}) \mid u' \in L^2(0,T;\mathcal{V}^*)\}$$

with inner product

$$(u,v)_{\mathcal{W}(\mathcal{V},\mathcal{V}^*)} = \int_0^T (u(t),v(t))_{\mathcal{V}} + \int_0^T (u'(t),v'(t))_{\mathcal{V}^*}$$

is a Hilbert space. Furthermore,

1. The embedding $\mathcal{W}(\mathcal{V},\mathcal{V}^*) \subset C([0,T];\mathcal{H})$ is continuous.
2. The embedding $\mathcal{D}([0,T];\mathcal{V}) \subset \mathcal{W}(\mathcal{V},\mathcal{V}^*)$ is dense.
3. For u, $v \in \mathcal{W}(\mathcal{V},\mathcal{V}^*)$, the map $t \mapsto (u(t),v(t))_{\mathcal{H}}$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}(u(t),v(t))_{\mathcal{H}} = \langle u'(t),v(t)\rangle_{\mathcal{V}^*,\mathcal{V}} + \langle u(t),v'(t)\rangle_{\mathcal{V},\mathcal{V}^*}$$

for almost every $t \in [0,T]$, hence the integration by parts formula

$$(u(T),v(T))_{\mathcal{H}} - (u(0),v(0))_{\mathcal{H}} = \int_0^T \langle u'(t),v(t)\rangle_{\mathcal{V}^*,\mathcal{V}} + \int_0^T \langle u(t),v'(t)\rangle_{\mathcal{V},\mathcal{V}^*}$$

holds.

Proof. The density result is Theorem 2.1 in [85, §1.2]. For the rest, consult Proposition 1.2 and Corollary 1.1 in [112, §III.1]. $\square$

We can characterise the weak derivative in terms of vector-valued test functions. This is useful because it more closely resembles the weak material derivative that we shall define later on.

Theorem 1.2.2 (Alternative characterisation of the weak derivative)**.** The weak derivative condition (1.3) is equivalent to

$$\int_0^T (u(t), \psi'(t))_{\mathcal{H}} = -\int_0^T \langle u'(t), \psi(t)\rangle_{\mathcal{V}^*,\mathcal{V}} \qquad \text{for all } \psi \in \mathcal{D}((0,T);\mathcal{V}).$$

We finish this subsection with some words on measurability.

Definition 1.2.3 (Strong measurability)**.** Let X be a Hilbert space. A function $f\colon [0,T] \to X$ is *strongly measurable* if there exists a sequence $\{f_n\}$ of simple functions such that $f_n(t) \to f(t)$ in X for almost every $t \in [0,T]$.

Definition 1.2.4 (Weak measurability)**.** Let X be a Hilbert space. A function $f\colon [0,T] \to X$ is *weakly measurable* if for every $x \in X$, the map $t \mapsto (f(t), x)_X$ is measurable on $[0,T]$.

Strong (or Bochner) measurability implies weak measurability. If the Hilbert space X turns out to be separable, then both notions of measurability are equivalent thanks to Pettis's theorem [108, §1.5, Theorem 1.34].

1.2.2 Evolving Hilbert spaces and the definition of L^2_H

Now we shall define Bochner-type function spaces to treat evolving spaces. We start with some notation and concepts on the evolution itself. We informally identify a family of Hilbert spaces $\{X(t)\}_{t\in[0,T]}$ with the symbol X, and given a family of maps $\phi_t\colon X_0 \to X(t)$ we define the following notion of **compatibility** of the pair $(X, (\phi_t)_{t\in[0,T]})$.

Definition 1.2.5 (Compatibility)**.** We say that a pair $(X, (\phi_t)_{t\in[0,T]})$ is *compatible* if all of the following conditions hold.

For each $t \in [0,T]$, $X(t)$ is a real separable Hilbert space (with $X_0 := X(0)$) and the map

$$\phi_t\colon X_0 \to X(t)$$

is a linear homeomorphism such that ϕ_0 is the identity. We denote by $\phi_{-t}\colon X(t) \to X_0$ the inverse of ϕ_t. Furthermore, we will assume that there exists a constant C_X independent of $t \in [0,T]$ such that

$$\begin{aligned} \|\phi_t u\|_{X(t)} &\leq C_X \|u\|_{X_0} & \forall u \in X_0 \\ \|\phi_{-t} u\|_{X_0} &\leq C_X \|u\|_{X(t)} & \forall u \in X(t). \end{aligned}$$

Finally, we assume that the map

$$t \mapsto \|\phi_t u\|_{X(t)}$$

is continuous for all $u \in X_0$.

We often write the pair as $(X, \phi_{(\cdot)})$ for convenience. We call ϕ_t and ϕ_{-t} the *pushforward* and *pullback* maps respectively. In the following we will assume compatibility of $(X, \phi_{(\cdot)})$. As a consequence of these assumptions, we have that the dual operator of ϕ_t, denoted $\phi_t^*\colon X^*(t) \to X_0^*$, is itself a linear homeomorphism, as is its inverse $\phi_{-t}^*\colon X_0^* \to X^*(t)$, and they satisfy

$$\begin{aligned} \|\phi_t^* f\|_{X_0^*} &\leq C_X \|f\|_{X^*(t)} \quad \forall f \in X^*(t) \\ \left\|\phi_{-t}^* f\right\|_{X^*(t)} &\leq C_X \|f\|_{X_0^*} \quad\quad \forall f \in X_0^*. \end{aligned}$$

By separability of X_0, it also follows that the map

$$t \mapsto \left\|\phi_{-t}^* f\right\|_{X^*(t)} \quad \forall f \in X_0^*$$

is measurable.

Remark 1.2.6. If we define $U(t,s)\colon X(s) \to X(t)$ by $U(t,s) := \phi_t\phi_{-s}$ for s, $t \in [0,T]$, it can be readily seen from $U(t,r)U(r,s) = \phi_t\phi_{-r}\phi_r\phi_{-s} = \phi_t\phi_{-s} = U(t,s)$ that the family of operators $U(t,s)$ is a two-parameter semigroup.

Remark 1.2.7. Note that the above implies the equivalence of norms

$$\begin{aligned} C_X^{-1}\|u\|_{X_0} \leq \|\phi_t u\|_{X(t)} \leq C_X \|u\|_{X_0} \quad\quad \forall u \in X_0, \\ C_X^{-1}\|f\|_{X^*(t)} \leq \|\phi_t^* f\|_{X_0^*} \leq C_X \|f\|_{X^*(t)} \quad \forall f \in X^*(t). \end{aligned}$$

We now define appropriate time-dependent function spaces to handle functions defined on evolving spaces. Our spaces are generalisations of those defined in [125].

Definition 1.2.8 (The spaces L^2_X and $L^2_{X^*}$)**.** Define the spaces

$$\begin{aligned} L^2_X &= \{u : [0,T] \to \bigcup_{t\in[0,T]} X(t) \times \{t\}, t \mapsto (\bar{u}(t), t) \mid \phi_{-(\cdot)}\bar{u}(\cdot) \in L^2(0,T;X_0)\} \\ L^2_{X^*} &= \{f : [0,T] \to \bigcup_{t\in[0,T]} X^*(t) \times \{t\}, t \mapsto (\bar{f}(t), t) \mid \phi_{(\cdot)}^*\bar{f}(\cdot) \in L^2(0,T;X_0^*)\}. \end{aligned}$$

More precisely, these spaces consist of equivalence classes of functions agreeing almost everywhere in $[0, T]$, just like ordinary Bochner spaces.

Remark 1.2.9. A few comments are in order.

1. We will generalise all of the results in this section to Banach spaces in §1.2.3. Nevertheless, since in this Hilbertian setting some proofs are easier (and more instructive), we choose not to start in the Banach space setting at the outset.

2. When using this framework for a particular application, one could start with a given family of Hilbert spaces $\{H(t)\}$ and then construct a family of maps $\{\phi_t\}$. On the other hand, one could begin with a reference space H_0 and a time-dependent mapping ϕ_t that is used to *define* $H(t)$. Of course, this needs to be done in way that ensures compatibility. Usually, the elliptic operator in a particular problem will dictate which time-dependent Hilbert spaces are appropriate.

3. If we are given a family $\{H(t)\}$, there is typically no unique way to define $\{\phi_t\}$; there may be many different maps that are compatible. See Remark 2.2.6 where we explain why this is useful when working on evolving domains or surfaces.

4. Following on from the previous comment, it is important to observe that the very definition of the space L^2_X depends on the mappings $\{\phi_t\}$.

We first show that the spaces in Definition 1.2.8 are inner product spaces, and later we prove that they are in fact Hilbert spaces. For $u \in L^2_X$, we will make an abuse of notation and identify $u(t) = (\bar{u}(t), t)$ with $\bar{u}(t)$ (and likewise for $f \in L^2_{X^*}$).

Theorem 1.2.10. The spaces L^2_X and $L^2_{X^*}$ are inner product spaces with the inner products

$$\begin{aligned} (u, v)_{L^2_X} &= \int_0^T (u(t), v(t))_{X(t)}\, \mathrm{d}t \\ (f, g)_{L^2_{X^*}} &= \int_0^T (f(t), g(t))_{X^*(t)}\, \mathrm{d}t. \end{aligned} \tag{1.4}$$

Proof. It is easy to verify that the expressions in (1.4) define inner products if the integrals on the right hand sides are well-defined, which we now check. For the L^2_X case, it suffices to show that $\|u(t)\|^2_{X(t)}$ is integrable for every $u \in L^2_X$. So let $u \in L^2_X$. Then $\tilde{u} := \phi_{-(\cdot)}u(\cdot) \in L^2(0, T; X_0)$. Define $F\colon [0, T] \times X_0 \to \mathbb{R}$ by $F(t, x) = \|\phi_t x\|_{X(t)}$. By assumption, $t \mapsto F(t, x)$ is measurable for all $x \in X_0$, and if $x_n \to x$ in X_0, then by the triangle inequality,

$$|F(t, x_n) - F(t, x)| \leq \|\phi_t(x_n - x)\|_{X(t)} \leq C_X \|x_n - x\|_{X_0} \to 0,$$

so $x \mapsto F(t, x)$ is continuous. Thus F is a Carathéodory function. Due to the condition $|F(t, x)| \leq C_X \|x\|_{X_0}$, by Remark 3.4.5 of [69] (see Theorem 1.2.11 below), the Nemytskii operator N_F defined by $(N_F x)(t) := F(t, x(t))$ maps $L^2(0, T; X_0) \to L^2(0, T)$, so that

$$\|N_F \tilde{u}\|^2_{L^2(0,T)} = \int_0^T \|u(t)\|^2_{X(t)} < \infty.$$

This proves the theorem for L^2_X. The process is the same for the case of $L^2_{X^*}$ except we replace ϕ_{-t} and ϕ_t with the dual maps ϕ^*_t and ϕ^*_{-t}. □

In the previous proof we made use of the following well-known result.

Theorem 1.2.11 (Remark 3.4.5 in [69])**.** Let X and Y be Banach spaces. If $f \colon \Omega \times X \to Y$ is a Carathéodory function and

$$\|f(\omega, x)\|_Y \leq a(\omega) + c \|x\|^{p/r}_X$$

holds for almost all $\omega \in \Omega$, where $a \in L^r(\Omega)$, $c > 0$, and $p, r \in [1, \infty)$, then the map $N_f \colon L^p(\Omega; X) \to L^r(\Omega; Y)$ defined by $(N_f(u))(\omega) := f(\omega, u(\omega))$ is continuous and bounded.

Lemma 1.2.12. Let $u \in L^2_X$ and $f \in L^2_{X^*}$. Then there exist simple measurable functions $u_n \in L^2(0, T; X_0)$ and $f_n \in L^2(0, T; X^*_0)$ such that for almost every $t \in [0, T]$,

$$\begin{aligned} \phi_t u_n(t) &\to u(t) && \text{in } X(t) \\ \phi^*_{-t} f_n(t) &\to f(t) && \text{in } X^*(t) \end{aligned}$$

as $n \to \infty$.

This lemma can be proved by using the density of simple measurable functions in $L^2(0, T; X_0)$. The following result is required to show that the above spaces are complete.

Lemma 1.2.13 (Isomorphism with standard Bochner spaces)**.** The maps

$$\begin{aligned} u &\mapsto \phi_{(\cdot)} u(\cdot) && \text{from } L^2(0, T; X_0) \text{ to } L^2_X \\ f &\mapsto \phi^*_{-(\cdot)} f(\cdot) && \text{from } L^2(0, T; X^*_0) \text{ to } L^2_{X^*} \end{aligned}$$

are both isomorphisms between the respective spaces.

For the proof of the L^2_X case, one makes an argument similar to that in the proof of Theorem 1.2.10 and shows that given an arbitrary $u \in L^2(0,T;X_0)$, the map $t \mapsto \|\phi_t u(t)\|^2_{X(t)}$ is indeed measurable (and then it follows that $\left\|\phi_{(\cdot)}u(\cdot)\right\|_{L^2_X}$ is finite). That the spaces are isomorphic follows from the above (which shows that there is a map from $L^2(0,T;X_0)$ to L^2_X) and the definition of L^2_X. The isomorphism is $T\colon L^2(0,T;X_0) \to L^2_X$ where

$$Tu = \phi_{(\cdot)}u(\cdot) \qquad \text{and} \qquad T^{-1}v = \phi_{-(\cdot)}v(\cdot).$$

It is easy to check that T is linear and bijective. The proof for the $L^2_{X^*}$ case uses the same readjustments as before.

The next lemma, which is a consequence of the uniform bounds on ϕ_t and ϕ^*_t, will be in constant use throughout this work.

Lemma 1.2.14. The equivalence of norms

$$\begin{aligned} \frac{1}{C_X}\|u\|_{L^2_X} &\leq \left\|\phi_{-(\cdot)}u(\cdot)\right\|_{L^2(0,T;X_0)} \leq C_X\|u\|_{L^2_X} && \forall u \in L^2_X \\ \frac{1}{C_X}\|f\|_{L^2_{X^*}} &\leq \left\|\phi^*_{(\cdot)}f(\cdot)\right\|_{L^2(0,T;X^*_0)} \leq C_X\|f\|_{L^2_{X^*}} && \forall f \in L^2_{X^*} \end{aligned}$$

holds.

Corollary 1.2.15. The spaces L^2_X and $L^2_{X^*}$ are separable Hilbert spaces.

Proof. Since L^2_X and $L^2(0,T;X_0)$ are isomorphic and the latter space is complete, so too is L^2_X by the equivalence of norms result in the previous lemma. The separability also follows from the previous lemma. ☐

We now investigate the relationship between the dual space of L^2_X and the space $L^2_{X^*}$. We in fact prove that these spaces can be identified; this requires the following preliminary lemmas.

Lemma 1.2.16. For $f \in L^2_{X^*}$ and $u \in L^2_X$, the map

$$t \mapsto \langle f(t), u(t)\rangle_{X^*(t),X(t)}$$

is integrable on $[0,T]$.

Proof. By considering the Carathéodory map $F\colon [0,T]\times X^*_0 \times X_0 \to \mathbb{R}$ defined by $F(t,x^*,x) = \langle \phi^*_{-t}x^*, \phi_t x\rangle_{X^*(t),X(t)}$ and using Remark 3.4.2 of [69] (see Theorem 1.2.17 below), given $f \in L^2_{X^*}$ and $u \in L^2_X$, we have with $\tilde{f} := \phi^*_{(\cdot)}f(\cdot)$ and $\tilde{u} := \phi_{-(\cdot)}u(\cdot)$ that the map $t \mapsto \langle \phi^*_{-t}\tilde{f}(t), \phi_t\tilde{u}(t)\rangle_{X^*(t),X(t)} = \langle f(t), u(t)\rangle_{X^*(t),X(t)}$ is measurable, since $t \mapsto \tilde{f}(t)$ and $t \mapsto \tilde{u}(t)$ are measurable. That the integral is finite is trivial. $\square$

Theorem 1.2.17 (Remark 3.4.2 in [69])**.** If X is separable metric space and Y is a metric space, then a Carathéodory function $f\colon \Omega \times X \to Y$ is jointly measurable, which implies that for every measurable function $u\colon \Omega \to X$, the function $\omega \mapsto f(\omega, u(\omega))$ is also measurable.

Lemma 1.2.18. Suppose that $f(t) \in X^*(t)$ for almost every $t \in [0,T]$ with

$$\int_0^T \|f(t)\|^2_{X^*(t)} < \infty,$$

and that for every $u \in L^2_X$, the map $t \mapsto \langle f(t), u(t)\rangle_{X^*(t),X(t)}$ is measurable. Then $f \in L^2_{X^*}$.

Proof. We have $\langle f(t), u(t)\rangle_{X^*(t),X(t)} = \langle \phi_t^* f(t), \phi_{-t}u(t)\rangle_{X_0^*,X_0}$, and the left hand side is measurable, hence the map

$$t \mapsto \langle \phi_t^* f(t), \phi_{-t}u(t)\rangle_{X_0^*,X_0}$$

is measurable on $[0,T]$ for every $u \in L^2_X$.

Given $w \in X_0$, the element $u(\cdot) := \phi_{(\cdot)}w \in L^2_X$, so we have (from Definition 1.2.4 or Footnote 80 in [107, §1.4, p. 36] for example) that $\phi^*_{(\cdot)}f(\cdot)\colon [0,T] \to X_0^*$ is weakly measurable. Now, as remarked after Definition 1.2.4, we use Pettis's theorem to conclude that $\phi^*_{(\cdot)}f(\cdot)$ is indeed strongly measurable. Hence we can compute

$$\left\|\phi^*_{(\cdot)}f(\cdot)\right\|^2_{L^2(0,T;X_0^*)} = \int_0^T \|\phi_t^* f(t)\|^2_{X_0^*} \leq C_X^2 \int_0^T \|f(t)\|^2_{X^*(t)} < \infty,$$

so $\phi^*_{(\cdot)}f(\cdot) \in L^2(0,T;X_0^*)$, giving $f \in L^2_{X^*}$. $\square$

Lemma 1.2.19 (Identification of $(L^2_X)^*$ and $L^2_{X^*}$)**.** The spaces $(L^2_X)^*$ and $L^2_{X^*}$ are isometrically isomorphic. Hence, we may identify $(L^2_X)^* \equiv L^2_{X^*}$, and the duality pairing of $f \in L^2_{X^*}$ with $u \in L^2_X$ is

$$\langle f, u\rangle_{L^2_{X^*},L^2_X} = \int_0^T \langle f(t), u(t)\rangle_{X^*(t),X(t)}\,\mathrm{d}t.$$

Proof. Define the linear map $\mathcal{J}\colon L^2_{X^*} \to (L^2_X)^*$ by

$$\langle \mathcal{J}f, \cdot\rangle_{(L^2_X)^*,L^2_X} = \int_0^T \langle f(t), (\cdot)(t)\rangle_{X^*(t),X(t)}\,\mathrm{d}t.$$

This is well-defined due to Lemma 1.2.16. We must check that $\mathcal{J}$ is an isometric isomorphism.

Suppose that $F \in (L^2_X)^*$. We first need to show that there exists a unique $f \in L^2_{X^*}$ such that $\mathcal{J}f = F$. To do this, we use the Riesz map $\mathcal{R}\colon (L^2_X)^* \to L^2_X$ to write

$$\langle F, u\rangle_{(L^2_X)^*, L^2_X} = (\mathcal{R}F, u)_{L^2_X} = \int_0^T (\mathcal{R}F(t), u(t))_{X(t)}, \tag{1.5}$$

and then with $\mathcal{S}_t^{-1}\colon X(t) \to X^*(t)$ denoting the inverse Riesz map on $X(t)$, we get

$$(\mathcal{R}F(t), u(t))_{X(t)} = \langle \mathcal{S}_t^{-1}(\mathcal{R}F(t)), u(t)\rangle_{X^*(t), X(t)}$$

for almost all $t \in [0, T]$. Now, from (1.5), the right hand side of this equality must be integrable. Hence

$$t \mapsto \langle \mathcal{S}_t^{-1}(\mathcal{R}F(t)), u(t)\rangle_{X^*(t), X(t)}$$

is measurable for every $u \in L^2_X$. Now, the question is whether $\mathcal{S}_{(\cdot)}^{-1}(\mathcal{R}F(\cdot)) \in L^2_{X^*}$. Clearly $\mathcal{S}_t^{-1}(\mathcal{R}F(t)) \in X^*(t)$, and by the isometry of the Riesz maps,

$$\int_0^T \left\|\mathcal{S}_t^{-1}(\mathcal{R}F(t))\right\|^2_{X^*(t)} = \int_0^T \|\mathcal{R}F(t)\|^2_{X(t)} = \|\mathcal{R}F\|^2_{L^2_X} = \|F\|^2_{(L^2_X)^*} \tag{1.6}$$

which is finite. Therefore, we obtain $\mathcal{S}_{(\cdot)}^{-1}(\mathcal{R}F(\cdot)) \in L^2_{X^*}$ by Lemma 1.2.18. So $\mathcal{J}(\mathcal{S}_{(\cdot)}^{-1}\mathcal{R}F(\cdot)) = F$.

For uniqueness, suppose that $\mathcal{J}f = 0$. Then

$$\begin{aligned}
\langle \mathcal{J}f, u\rangle_{(L^2_X)^*, L^2_X} &= \int_0^T \langle f(t), u(t)\rangle_{X^*(t), X(t)} \\
&= \int_0^T \langle \phi_t^* f(t), \phi_{-t}u(t)\rangle_{X_0^*, X_0} \\
&= \langle \phi_{(\cdot)}^* f(\cdot), \hat{u}\rangle_{L^2(0,T;X_0^*), L^2(0,T;X_0)}, \qquad \text{(with } \hat{u} = \phi_{-(\cdot)}u(\cdot)\text{)}
\end{aligned}$$

which holds for all $\hat{u} \in L^2(0, T; X_0)$. This implies that $f = 0$.

To see that $\mathcal{J}$ is an isometry, we define $\mathcal{J}^{-1}\colon (L^2_X)^* \to L^2_{X^*}$ by $\mathcal{J}^{-1}F = \mathcal{S}_{(\cdot)}^{-1}\mathcal{R}F(\cdot)$ and use (1.6) to conclude. □

Although we have no notion of continuity in time for a function $u \in L^2_X$, we can nevertheless make the following definition.

Definition 1.2.20 (Spaces of pushed-forward continuously differentiable functions).

Define

$$\begin{aligned}
C_X^k &= \{\xi \in L_X^2 \mid \phi_{-(\cdot)}\xi(\cdot) \in C^k([0,T]; X_0)\} \quad \text{for } k \in \{0, 1, ...\} \\
\mathcal{D}_X(0,T) &= \{\eta \in L_X^2 \mid \phi_{-(\cdot)}\eta(\cdot) \in \mathcal{D}((0,T); X_0)\} \\
\mathcal{D}_X[0,T] &= \{\eta \in L_X^2 \mid \phi_{-(\cdot)}\eta(\cdot) \in \mathcal{D}([0,T]; X_0)\}.
\end{aligned}$$

Since $\mathcal{D}((0,T); X_0) \subset \mathcal{D}([0,T]; X_0)$, we have

$$\mathcal{D}_X(0,T) \subset \mathcal{D}_X[0,T] \subset C_X^k.$$

We sometimes write $\mathcal{D}_X$ instead of $\mathcal{D}_X(0,T)$.

1.2.3 Evolving Banach spaces and the definition of L_X^p

When we work on nonlinear problems in Chapters 3 and 4, we will need a generalisation of the theory in §1.2.2 to Banach spaces. Let us now define L_X^p and study some of its properties.

For each $t \in [0,T]$, let $X(t)$ be a real Banach space with $X_0 := X(0)$. We informally identify the family $\{X(t)\}_{t\in[0,T]}$ with the symbol X. Let there be a linear homeomorphism $\phi_t \colon X_0 \to X(t)$ for each $t \in [0,T]$ (with the inverse $\phi_{-t} \colon X(t) \to X_0$) such that ϕ_0 is the identity. We assume that there exists a constant C_X independent of $t \in [0,T]$ such that

$$\begin{aligned}
\|\phi_t u\|_{X(t)} &\leq C_X \|u\|_{X_0} \quad && \forall u \in X_0 \\
\|\phi_{-t} u\|_{X_0} &\leq C_X \|u\|_{X(t)} \quad && \forall u \in X(t).
\end{aligned} \tag{1.7}$$

We assume for all $u \in X_0$ that the map $t \mapsto \|\phi_t u\|_{X(t)}$ is measurable.

Definition 1.2.21. Define the Banach spaces

$$L_X^p = \{u : [0,T] \to \bigcup_{t\in[0,T]} X(t) \times \{t\}, \quad t \mapsto (\hat{u}(t), t) \quad \mid \quad \phi_{-(\cdot)}\hat{u}(\cdot) \in L^p(0,T; X_0)\}$$

for $p \in [1, \infty)$, and

$$L_X^\infty = \{u \in L_X^2 \mid \operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_{X(t)} < \infty\}$$

endowed with the norm

$$\|u\|_{L^p_X} = \begin{cases} \left(\int_0^T \|u(t)\|^p_{X(t)}\right)^{\frac{1}{p}} & \text{for } p \in [1,\infty) \\ \operatorname{ess\,sup}_{t\in[0,T]} \|u(t)\|_{X(t)} & \text{for } p = \infty. \end{cases} \tag{1.8}$$

As before, we made an abuse of notation after the definition of the first space and identified $u(t) = (\hat{u}(t), t)$ with $\hat{u}(t)$. That (1.8) defines a norm is easy to see once one checks that the integrals are well-defined (the case $p = \infty$ is easy), which can be shown by a straightforward adaptation of the proof of Theorem 1.2.10 for the case when each $X(t)$ is separable and the proof of Lemma 3.5 in [125] for the non-separable case. The fact that L^p_X is a Banach space follows from Lemma 1.2.23 below.

Notation 1.2.22. Given a function $u \in L^p_X$, the notation $\tilde{u}$ will be used to mean the pullback $\tilde{u}(\cdot) := \phi_{-(\cdot)}u(\cdot) \in L^p(0,T;X_0)$, and vice-versa.

Lemma 1.2.23. The spaces $L^p(0,T;X_0)$ and L^p_X are isomorphic via $\phi_{(\cdot)}$ with an equivalence of norms:

$$\frac{1}{C_X}\|u\|_{L^p_X} \leq \left\|\phi_{-(\cdot)}u(\cdot)\right\|_{L^p(0,T;X_0)} \leq C_X\|u\|_{L^p_X} \quad \text{for all } u \in L^p_X.$$

Proof. First, suppose that $p \in [1,\infty)$. We show that if $u \in L^p(0,T;X_0)$, then $\phi_{(\cdot)}u(\cdot) \in L^p_X$.

Let $u \in L^p(0,T;X_0)$ be arbitrary. By density, there exists a sequence of simple functions $u_n \in L^p(0,T;X_0)$ with

$$\|u_n - u\|_{L^p(0,T;X_0)} \to 0$$

and thus for almost every t,

$$\|u_n(t) - u(t)\|_{X_0} \to 0$$

for a subsequence, which we relabelled. We have that $\phi_t u_n(t) \to \phi_t u(t)$ in $X(t)$ by continuity; this implies

$$\|\phi_t u_n(t)\|_{X(t)} \to \|\phi_t u(t)\|_{X(t)} \qquad \text{pointwise a.e.} \tag{1.9}$$

Write $u_n(t) = \sum_{i=1}^{M_n} u_{n,i}\mathbf{1}_{B_i}(t)$ where the $u_{n,i} \in X_0$ and the B_i are measurable, disjoint and partition $[0,T]$. Then

$$\phi_t u_n(t) = \sum_{i=1}^{M_n} \phi_t(u_{n,i})\mathbf{1}_{B_i}(t) \in X(t).$$

Taking norms and exponentiating, we get

$$\|\phi_t u_n(t)\|_{X(t)}^p = \sum_{i=1}^{M_n} \|\phi_t u_{n,i}\|_{X(t)}^p \, \mathbf{1}_{B_i}^p(t),$$

which is measurable (with respect to t) since, by assumption, the $\|\phi_t u_{n,i}\|_{X(t)}$ are continuous and a finite sum of measurable functions is measurable. Thus, by (1.9), $\|\phi_t u(t)\|_{X(t)}$, is measurable. Finally,

$$\int_0^T \|\phi_t u(t)\|_{X(t)}^p \leq \int_0^T C_X^p \, \|u(t)\|_{X_0}^p = C_X^p \, \|u\|_{L^p(0,T;X_0)}^p \, ,$$

so $\phi_{(\cdot)}u(\cdot) \in L^p_X$.

So there is a map from $L^p(0,T;X_0)$ to L^p_X and vice-versa from the definition of L^p_X. The isomorphism between the spaces is $T\colon L^p(0,T;X_0) \to L^p_X$ where

$$Tu = \phi_{(\cdot)}u(\cdot), \quad \text{and} \quad T^{-1}v = \phi_{-(\cdot)}v(\cdot).$$

It is easy to check that T is linear and bijective. The equivalence of norms follows by the bounds on $\phi_{-t}\colon X(t) \to X_0$

$$\frac{1}{C_X} \|u(t)\|_{X(t)} \leq \|\phi_{-t}u(t)\|_{X_0} \leq C_X \|u(t)\|_{X(t)} \, .$$

Now let $p = \infty$. Let $u \in L^\infty_X$. Measurability of $\tilde{u}$ follows since $u \in L^2_X$. Now, by definition, we have that for all $t \in [0,T]\backslash N$, $\|u(t)\|_{X(t)} \leq A$ where N is a null set and $A = \|u\|_{L^\infty_X}$. This means that for all $t \in [0,T]\backslash N$, $C_X^{-1} \|\tilde{u}(t)\|_{X_0} \leq \|u(t)\|_{X(t)} \leq A$ by the assumption (1.7), i.e.,

$$\|\tilde{u}\|_{L^\infty(0,T;X_0)} = \operatorname*{ess\,sup}_{t\in[0,T]} \|\tilde{u}(t)\|_{X_0} \leq C_X A = C_X \|u\|_{L^\infty_X},$$

so $\tilde{u} \in L^\infty(0,T;X_0)$. Similarly, we conclude that if $\tilde{u} \in L^\infty(0,T;X_0)$ then $u \in L^\infty_X$. $\square$

Remark 1.2.24. The dual operator $\phi^*_{-t}\colon X_0^* \to X^*(t)$ is also a linear homeomorphism with $\left\|\phi^*_{-t}\right\| = \|\phi_{-t}\|$ and $(\phi^*_{-t})^{-1} = \phi^*_t$ [80, Theorem 4.5-2 and §4.5], and if

X_0 is separable, $t \mapsto \left\|\phi^*_{-t} f\right\|_{X^*(t)}$ is measurable for $f \in X_0^*$; thus, in the separable setting, the dual operator also satisfies the same boundedness properties as ϕ_t. This means that the spaces $L^p_{X^*}$ are also well-defined Banach spaces given separable $\{X(t)\}_{t\in[0,T]}$ (the map $\phi^*_{-(\cdot)}$ plays the same role as $\phi_{(\cdot)}$ did for the spaces L^p_X).

Dual spaces

In this subsection, we assume that $\{X(t)\}_{t\in[0,T]}$ is reflexive. In order to retrieve weakly convergent subsequences from sequences that are bounded in L^p_X, we need L^p_X to be reflexive. This leads us to consider a characterisation of the dual spaces. We let $p \in [1, \infty)$ and (p, q) be a conjugate pair in this section.

Theorem 1.2.25. The space $(L^p_X)^*$ is isometrically isomorphic to $L^q_{X^*}$, and hence we may identify $(L^p_X)^* \equiv L^q_{X^*}$ and the duality pairing of $f \in L^q_{X^*}$ with $u \in L^p_X$ is given by

$$\langle f, u\rangle_{L^q_{X^*}, L^p_X} = \int_0^T \langle f(t), u(t)\rangle_{X^*(t), X(t)}.$$

To prove this theorem, although we can exploit the fact that the pullback is in a Bochner space, showing that the natural duality map is isometric is not so straightforward because $\phi_{(\cdot)}$ is not assumed to be an isometry. In fact, we have to go back to the foundations and emulate the proof for the dual space identification for Bochner spaces; see [46, §IV].

Lemma 1.2.26. For every $g \in L^q_{X^*}$, the expression

$$l(f) = \int_0^T \langle g(t), f(t)\rangle_{X^*(t), X(t)} \qquad \text{for all } f \in L^p_X \tag{1.10}$$

defines a functional $l \in (L^p_X)^*$ such that $\|l\| = \|g\|_{L^q_{X^*}}$.

Proof. Let $g \in L^q_{X^*}$ and define $l \colon L^p_X \to \mathbb{R}$ by (1.10); the integral is well-defined by similar reasoning as before (see Lemma 1.2.16). By Hölder's inequality, we have $|l(f)| \le \|g\|_{L^q_{X^*}} \|f\|_{L^p_X}$, so $l \in (L^p_X)^*$ and $\|l\| \le \|g\|_{L^q_{X^*}}$. We now show the reverse inequality. First suppose g has the form $g(t) = \sum x^*_{i,t} \chi_{E_i}(t)$ where the $x^*_{i,t} \in X^*(t)$ and the E_i are measurable, pairwise disjoint and partition $[0, T]$. It is clear that $\|g(t)\|_{X^*(t)} = \sum \left\|x^*_{i,t}\right\|_{X^*(t)} \chi_{E_i}(t)$. Let $h(t) = \|g(t)\|^{q/p}_{X^*(t)} / \|g\|^{q/p}_{L^q_{X^*}}$ which satisfies $\|h\|^p_{L^p(0,T)} = 1$ and $\int_0^T \|g(t)\|_{X^*(t)} h(t) = \|g\|_{L^q_{X^*}}$, hence for any $\epsilon > 0$ we have

$$\int_0^T \|g(t)\|_{X^*(t)} h(t) \ge \|g\|_{L^q_{X^*}} - \frac{\epsilon}{2}. \tag{1.11}$$

Now choose $x_{i,t} \in X(t)$, $\|x_{i,t}\|_{X(t)} = 1$ such that

$$\|x^*_{i,t}\|_{X^*(t)} - \langle x^*_{i,t}, x_{i,t}\rangle_{X^*(t),X(t)} \le \frac{\epsilon}{2\|h\|_{L^1(0,T)}}. \tag{1.12}$$

Define $f \in L^p_X$ by $f(t) = \sum x_{i,t}h(t)\chi_{E_i}(t)$ and note that $\|f\|^p_{L^p_X} = \|h\|^p_{L^p(0,T)}$. We obtain using (1.12) and (1.11) that $l(f) \ge \|g\|_{L^q_{X^*}} - \epsilon$. This proves that $\|l\| = \|g\|_{L^q_{X^*}}$ whenever $g(t) = \sum x^*_{i,t}\chi_{E_i}(t)$ is of the stated form. Now suppose $g \in L^q_{X^*}$ is arbitrary. Then there exist $\tilde{g}_n(t) = \sum \tilde{g}_{i,n}\chi_{E_i}(t)$ with $\tilde{g}_{i,n} \in X^*_0$ such that $\tilde{g}_n \to \tilde{g}$ in $L^q(0,T;X^*_0)$ and so the sequence $g_n(t) := \phi^*_{-t}\tilde{g}_n(t) = \sum \phi^*_{-t}\tilde{g}_{i,n}\chi_{E_i}(t)$ satisfies $g_n \to g$ in $L^q_{X^*}$. Because the $\phi^*_{-t}\tilde{g}_{i,n} \in X^*(t)$, we know by our efforts above that $l_n\colon L^p_X \to \mathbb{R}$ defined $l_n(f) = \int_0^T \langle g_n(t), f(t)\rangle_{X^*(t),X(t)}$ has norm $\|l_n\| = \|g_n\|_{L^q_{X^*}}$. We also have

$$\|l_n - l\| \le \|g_n - g\|_{L^q_{X^*}} \to 0$$

which implies $\lim_{n\to\infty}\|l_n\| = \|l\|$ and also $\lim_{n\to\infty}\|l_n\| = \lim_{n\to\infty}\|g_n\|_{L^q_{X^*}} = \|g\|_{L^q_{X^*}}$. □

We have shown that $\mathcal{J}\colon L^q_{X^*} \to (L^p_X)^*$ defined by the map $\mathcal{J}(g) := l(\cdot) = \int_0^T \langle g(t), (\cdot)(t)\rangle_{X^*(t),X(t)}$ is isometric: $\|\mathcal{J}g\|_{(L^q_X)^*} = \|l\| = \|g\|_{L^q_{X^*}}$. We now show that $\mathcal{J}$ is onto. Given $l \in (L^p_X)^*$, define $\tilde{L}\colon L^p(0,T;X_0) \to \mathbb{R}$ by $\tilde{L}(\tilde{v}) = l(\phi_{(\cdot)}\tilde{v}(\cdot)) = l(v)$ for all $\tilde{v} \in L^p(0,T;X_0)$. It is obvious that $\tilde{L} \in L^p(0,T;X_0)^*$, and by the dual space identification for Bochner spaces, there exists an $\tilde{L}^* \in L^q(0,T;X^*_0)$ such that

$$\langle l, v\rangle_{(L^p_X)^*,L^p_X} = \langle \tilde{L}, \tilde{v}\rangle_{L^p(0,T;X_0)^*,L^p(0,T;X_0)} = \int_0^T \langle \phi^*_{-t}\tilde{L}^*(t), v(t)\rangle_{X^*(t),X(t)},$$

so $\mathcal{J}(\phi^*_{-(\cdot)}\tilde{L}^*(\cdot)) = l$ where $\phi^*_{-(\cdot)}\tilde{L}^*(\cdot) \in L^q_{X^*}$. Hence $\mathcal{J}$ is onto, and we have proved Theorem 1.2.25.

1.2.4 Evolving Hilbert space structure for parabolic equations

In the preceding, we set up a Hilbert space L^2_X and its dual $L^2_{X^*}$ based on an arbitrary family of separable Hilbert spaces $\{X(t)\}_{t\in[0,T]}$ and a suitable family of maps $\{\phi_t\}_{t\in[0,T]}$. We now lay the groundwork for posing PDEs on evolving spaces. For each $t \in [0,T]$, let $V(t)$ and $H(t)$ be (real) separable Hilbert spaces with $V_0 := V(0)$ and $H_0 := H(0)$ such that $V_0 \subset H_0$ is a continuous and dense embedding. Identifying H_0 with its dual space H^*_0, it follows that $H_0 \subset V^*_0$ is also continuous and dense. In other words, $V_0 \subset H_0 \subset V^*_0$ is a Gelfand or evolution triple of Hilbert spaces (i.e., a Hilbert triple) [108, §7.2].

Assumptions 1.2.27. We will assume compatibility in the sense of Definition 1.2.5 for the family $\{H(t)\}_{t\in[0,T]}$ and a family of linear homeomorphisms $\{\phi_t\}_{t\in[0,T]}$; that is, we assume $(H,\phi_{(\cdot)})$ is a compatible pair. In addition, we also assume that $(V,\phi_{(\cdot)}|_{V_0})$ is compatible. We will simply write ϕ_t instead of $\phi_t|_{V_0}$, and we will denote the dual operator of $\phi_t\colon V_0\to V(t)$ by $\phi_t^*\colon V^*(t)\to V_0^*$; we are not interested in the dual of $\phi_t\colon H_0\to H(t)$.

It then follows that for each $t\in[0,T]$, $V(t)\subset H(t)$ is continuously and densely embedded. Let us summarise the meaning and consequences of Assumptions 1.2.27 for the convenience of the reader.

1. For each $t\in[0,T]$, there exists a linear homeomorphism
$$\phi_t\colon H_0\to H(t)$$
such that ϕ_0 is the identity.
2. The restriction $\phi_t|_{V_0}$ (which we will denote by ϕ_t) is also a linear homeomorphism from V_0 to $V(t)$.
3. There exist constants C_H and C_V independent of $t\in[0,T]$ such that
$$\begin{aligned}\|\phi_t u\|_{H(t)} &\le C_H\|u\|_{H_0} && \forall u\in H_0,\\ \|\phi_t u\|_{V(t)} &\le C_V\|u\|_{V_0} && \forall u\in V_0.\end{aligned}$$
4. We will only be interested in the dual operator of $\phi_t\colon V_0\to V(t)$, denoted by $\phi_t^*\colon V^*(t)\to V_0^*$, which satisfies
$$\|\phi_t^* f\|_{V_0^*}\le C_V\|f\|_{V^*(t)} \qquad \forall f\in V^*(t).$$
5. The inverses of ϕ_t and ϕ_t^* will be denoted by ϕ_{-t} and ϕ_{-t}^* respectively, and these are uniformly bounded:
$$\begin{aligned}\|\phi_{-t}u\|_{H_0} &\le \tilde{C}_H\|u\|_{H(t)} && \forall u\in H(t),\\ \|\phi_{-t}u\|_{V_0} &\le \tilde{C}_V\|u\|_{V(t)} && \forall u\in V(t),\\ \left\|\phi_{-t}^* f\right\|_{V^*(t)} &\le \tilde{C}_V\|f\|_{V_0^*} && \forall f\in V_0^*.\end{aligned}$$

6. The maps

$$t \mapsto \|\phi_t u\|_{H(t)} \qquad \forall u \in H_0$$
$$t \mapsto \|\phi_t u\|_{V(t)} \qquad \forall u \in V_0$$

are continuous, and the map

$$t \mapsto \left\|\phi_{-t}^* f\right\|_{V^*(t)} \qquad \forall f \in V_0^*$$

is measurable.

Our work in §1.2.2 tells us that the spaces L_H^2, L_V^2, and $L_{V^*}^2$ are Hilbert spaces with the inner product given by the formula (1.4).

Remark 1.2.28. These homeomorphisms ϕ_t are similar to Arbitrary Lagrangian Eulerian (ALE) maps that are ubiquitous in applications on moving domains. See [5] for an account of the ALE framework and a comparable set-up.

By the density of $L^2(0,T;V_0)$ in $L^2(0,T;H_0)$, we obtain the next result.

Lemma 1.2.29. The embedding $L_V^2 \subset L_H^2$ is continuous and dense.

Identifying L_H^2 with $L_{H^*}^2$ in the natural manner, we have that $L_V^2 \subset L_H^2 \subset L_{V^*}^2$ is a Hilbert triple. We make use of the formula

$$\langle f, u\rangle_{L_{V^*}^2, L_V^2} = (f,u)_{L_H^2} \quad \text{whenever } f \in L_H^2 \text{ and } u \in L_V^2.$$

1.2.5 Abstract strong and weak material derivatives

Suppose $\{\Gamma(t)\}_{t\in[0,T]}$ is a family of (sufficiently smooth) hypersurfaces evolving with velocity field $\mathbf{w}$, and that for each $t \in [0,T]$, $u(t)$ is a sufficiently smooth function defined on $\Gamma(t)$. Then the appropriate time derivative of u *takes into account the movement of the spatial points too*, and this time derivative is known as the (strong) *material derivative*, which we can write informally as

$$\dot{u}(t,x) = \frac{d}{dt}u(t,x(t)) = u_t(t,x) + \nabla u(t,x)\cdot \mathbf{w}(t,x). \tag{1.13}$$

This is well-studied: see [33] or [35, §1.2] for the flat case. Our aim is to generalise this material derivative to arbitrary functions and arbitrary evolving spaces (and not just merely evolving surfaces).

Definition 1.2.30 (Strong material derivative)**.** For $\xi \in C^1_X$ define the *strong material derivative* $\dot{\xi} \in C^0_X$ by

$$\dot{\xi}(t) := \phi_t \left(\frac{d}{dt}(\phi_{-t}\xi(t)) \right). \tag{1.14}$$

This definition is generalised from [125]. So we see that the space C^1_X is the space of functions with a strong material derivative, justifying the notation. In the evolving surface case, we show in Chapter 2 that this abstract formula agrees with (1.13). Note that the strong material derivative on moving domains is sometimes also called the Lagrangian derivative.

Remark 1.2.31. Of course, the formula (1.13) is sensible for functions $\xi \in C^0_X$ such that $\phi_{-(\cdot)}\xi(\cdot)$ is differentiable, but to avoid too clumsy a notation for such sets of functions we leave this observation as a remark.

The following remark observes that the pushforward of elements of X_0 into $X(t)$ have zero material derivative.

Remark 1.2.32. Observe that given $\eta \in X_0$,

$$\dot{(\phi_t \eta)} = 0$$

and that for $\xi \in C^1_X$

$$\dot{\xi} = 0 \iff \exists\ \eta \in X_0 \ \text{ such that } \ \xi(t) = \phi_t \eta.$$

It may be the case that solutions to the PDE (1.2)

$$\dot{u}(t) + A(t)u(t) = f(t)$$

may not exist if we ask for $u \in C^1_V$, that is, they may not possess strong material derivatives. We can relax this and ask for $\dot{u}$ to exist in a weaker sense, just like one does for the usual time derivative in parabolic problems on fixed domains. Heuristically, what should such a weak material derivative satisfy? Taking a clue from Lemma 1.2.1, we expect

$$\frac{d}{dt}(u(t), v(t))_{H(t)} = \langle \dot{u}(t), v(t) \rangle_{V^*(t),V(t)} + \langle \dot{v}(t), u(t) \rangle_{V^*(t),V(t)} + \text{extra term}$$

where we envisage an extra term because the Hilbert space associated with the inner product depends on t itself, and certainly we should require the integration by parts formula

$$\int_0^T \frac{d}{dt}(u(t), \eta(t))_{H(t)} = 0 \quad \forall \eta \in \mathcal{D}_V(0,T).$$

The identification of this extra term and a definition of the weak material derivative is what the rest of this section is devoted to. Observe that, formally, the method to obtain this extra term is to rewrite the $H(t)$ inner product as a bilinear form on H_0, differentiate in time and then pushforward again onto $H(t)$. This is the process we now repeat rigorously.

Definition 1.2.33 (Relationship between the inner product on $H(t)$ and the space H_0)**.** For all $t \in [0,T]$, define the bounded bilinear form $\hat{b}(t;\cdot,\cdot)\colon H_0 \times H_0 \to \mathbb{R}$ by

$$\hat{b}(t; u_0, v_0) = (\phi_t u_0, \phi_t v_0)_{H(t)} \quad \forall u_0,\, v_0 \in H_0.$$

This gives us a way of pulling back the inner product on $H(t)$ onto a bilinear form on H_0 by the formula $(u,v)_{H(t)} = \hat{b}(t; \phi_{-t}u, \phi_{-t}v)$. It is also clear that $\hat{b}(0;\cdot,\cdot) = (\cdot,\cdot)_{H_0}$ by definition. In fact, one can see for each $t \in [0,T]$ that $\hat{b}(t;\cdot,\cdot)$ is an inner product on H_0 (and it is norm-equivalent with the norm on H_0); thanks to the Riesz representation theorem, there exists for each $t \in [0,T]$ a bounded linear operator $T_t\colon H_0 \to H_0$ such that

$$\hat{b}(t; u_0, v_0) = (T_t u_0, v_0)_{H_0} = (u_0, T_t v_0)_{H_0}. \tag{1.15}$$

Remark 1.2.34. It is not difficult to see that $T_t \equiv \phi_t^A \phi_t$, where $\phi_t^A\colon H(t) \to H_0$ denotes the Hilbert-adjoint of $\phi_t\colon H_0 \to H(t)$.

Now that the inner product on $H(t)$ has a representation as a bilinear form $\hat{b}(t;\cdot,\cdot)$ on H_0, we would like to be able to differentiate $\hat{b}(t;\cdot,\cdot)$.

Assumptions 1.2.35. We shall assume the following for all $u_0,\, v_0 \in H_0$:

$$\theta(t, u_0) := \frac{d}{dt}\left\|\phi_t u_0\right\|^2_{H(t)} \text{ exists classically} \tag{1.16}$$

$$u_0 \mapsto \theta(t, u_0) \text{ is continuous} \tag{1.17}$$

$$|\theta(t, u_0 + v_0) - \theta(t, u_0 - v_0)| \leq C \left\|u_0\right\|_{H_0} \left\|v_0\right\|_{H_0} \tag{1.18}$$

where the constant C is independent of $t \in [0,T]$.

We are now able to define $\hat{\lambda}(t;\cdot,\cdot)\colon H_0 \times H_0 \to \mathbb{R}$ by

$$\begin{aligned}
\hat{\lambda}(t;u_0,v_0) &:= \frac{d}{dt}\hat{b}(t;u_0,v_0) \\
&= \frac{1}{4}\frac{d}{dt}(\|\phi_t u_0 + \phi_t v_0\|^2_{H(t)} - \|\phi_t u_0 - \phi_t v_0\|^2_{H(t)}) \\
&= \frac{1}{4}\left(\theta(t,u_0+v_0) - \theta(t,u_0-v_0)\right)
\end{aligned} \tag{1.19}$$

where we used the polarisation identity for the second equality. Denoting by $\hat{\Lambda}(t)$ the operator

$$\langle \hat{\Lambda}(t)u_0, v_0\rangle := \hat{\lambda}(t;u_0,v_0), \tag{1.20}$$

it follows by (1.18) that $\hat{\Lambda}(t)\colon H_0 \to H_0^*$.

Definition 1.2.36 (The bilinear form $\lambda(t;\cdot,\cdot)$)**.** For u, $v \in H(t)$, define the bilinear form $\lambda(t;\cdot,\cdot)\colon H(t)\times H(t) \to \mathbb{R}$ by

$$\lambda(t;u,v) = \hat{\lambda}(t;\phi_{-t}u, \phi_{-t}v).$$

Remark 1.2.37. This form $\lambda(t,\cdot,\cdot)$ can be thought of as the map that arises from differentiating the time-dependence in the $H(t)$ inner product. It is related to differentiating the metric in a Riemannian manifold.

Lemma 1.2.38. For all u, $v \in L^2_H$, the map $t \mapsto \lambda(t;u(t),v(t))$ is measurable and $\lambda(t;\cdot,\cdot)\colon H(t)\times H(t) \to \mathbb{R}$ is bounded independently of t:

$$|\lambda(t;u,v)| \leq C\,\|u\|_{H(t)}\,\|v\|_{H(t)}\,.$$

Proof. If $u, v \in L^2_H$, then by (1.19),

$$\begin{aligned}
\lambda(t;u(t),v(t)) &= \hat{\lambda}(t;\phi_{-t}u(t),\phi_{-t}v(t)) \\
&= \frac{1}{4}\left(\theta(t,\phi_{-t}u(t)+\phi_{-t}v(t)) - \theta(t,\phi_{-t}u(t)-\phi_{-t}v(t))\right),
\end{aligned}$$

and it follows that $t \mapsto \lambda(t;u(t),v(t))$ is measurable because $t \mapsto \theta(t,\phi_{-t}w(t))$ is measurable for $w \in L^2_H$. This in turn can be seen by noticing that $\theta\colon [0,T]\times H_0 \to \mathbb{R}$ is a Carathéodory function: the map $t \mapsto \theta(t,x)$ is measurable and by assumption (1.17) the map $x \mapsto \theta(t,x)$ is continuous; thus by [69, Remark 3.4.2] (see Theorem 1.2.17) the desired measurability is achieved. The bound on $\lambda(t;\cdot,\cdot)$ is a consequence of the assumption (1.18). $\square$

Lemma 1.2.39. For σ_1, $\sigma_2 \in C^1([0,T];H_0)$, the map $t \mapsto \hat{b}(t;\sigma_1(t),\sigma_2(t))$ is differentiable in the classical sense and

$$\frac{d}{dt}\hat{b}(t;\sigma_1(t),\sigma_2(t)) = \hat{b}(t;\sigma_1'(t),\sigma_2(t)) + \hat{b}(t;\sigma_1(t),\sigma_2'(t)) + \hat{\lambda}(t;\sigma_1(t),\sigma_2(t)).$$

This follows simply by using the definition of the derivative as a limit.

Definition 1.2.40 (Weak material derivative)**.** For $u \in L^2_V$, if there exists a function $g \in L^2_{V^*}$ such that

$$\int_0^T \langle g(t), \eta(t)\rangle_{V^*(t),V(t)} = -\int_0^T (u(t),\dot{\eta}(t))_{H(t)} - \int_0^T \lambda(t;u(t),\eta(t))$$

holds for all $\eta \in \mathcal{D}_V(0,T)$, then we say that g is the *weak material derivative* of u, and we write

$$\dot{u} = g \quad \text{or} \quad \partial^\bullet u = g.$$

Remark 1.2.41. The notation $\dot{u}$ is clean but inelegant when taking the material derivative of products or compositions, for which $\partial^\bullet(uv)$ or $\partial^\bullet f(u)$ is better. The latter notation is also well established in the literature, see for example [9, 59].

This concept of a weak material derivative is indeed well-defined: if it exists, it is unique, and every strong material derivative is also a weak material derivative. It is easy to prove these facts: for uniqueness, assume there exist two material derivatives for the same function and then linearity and the density of $\mathcal{D}((0,T);V_0)$ (the space of test functions) in $L^2(0,T;V_0)$ gives the result. To show that a strong material derivative is also a weak material derivative, one can use Lemma 1.2.39 and the relations between $\hat{b}(t;\cdot,\cdot)$, $\hat{\lambda}(t;\cdot,\cdot)$, and $\lambda(t;\cdot,\cdot)$.

1.2.6 Solution space

We can now consider the spaces that solutions of our PDEs will lie in.

Definition 1.2.42 (The space $W(V,V^*)$)**.** Define the solution space

$$W(V,V^*) = \{u \in L^2_V \mid \dot{u} \in L^2_{V^*}\}$$

and endow it with the inner product

$$(u,v)_{W(V,V^*)} = \int_0^T (u(t),v(t))_{V(t)} + \int_0^T (\dot{u}(t),\dot{v}(t))_{V^*(t)}.$$

In order to prove existence theorems, we need some properties of the space $W(V, V^*)$ which turns out to be deeply linked with the following standard Sobolev–Bochner space.

Definition 1.2.43 (The space $\mathcal{W}(V_0, V_0^*)$)**.** Define

$$\mathcal{W}(V_0, V_0^*) = \{v \in L^2(0, T; V_0) \mid v' \in L^2(0, T; V_0^*)\}$$

to be the space $\mathcal{W}(\mathcal{V}, \mathcal{V}^*)$ introduced in §1.2.1 with the Hilbert triple setting $\mathcal{V} = V_0$ and $\mathcal{H} = H_0$ (recall that $\mathcal{W}(\mathcal{V}, \mathcal{V}^*)$ is a notation that hides the pivot space $\mathcal{H}$, which is used to define it).

It is convenient to introduce the following notion of *evolving space equivalence* which will enable us to transfer many essential properties of $\mathcal{W}(V_0, V_0^*)$ to the space $W(V, V^*)$.

Assumption and Definition 1.2.44. We assume that there is an *evolving space equivalence* between $W(V, V^*)$ and $\mathcal{W}(V_0, V_0^*)$. By this we mean that

$$v \in W(V, V^*) \quad \text{if and only if} \quad \phi_{-(\cdot)}v(\cdot) \in \mathcal{W}(V_0, V_0^*),$$

and the equivalence of norms

$$C_1 \left\|\phi_{-(\cdot)}v(\cdot)\right\|_{\mathcal{W}(V_0,V_0^*)} \leq \|v\|_{W(V,V^*)} \leq C_2 \left\|\phi_{-(\cdot)}v(\cdot)\right\|_{\mathcal{W}(V_0,V_0^*)}$$

holds.

Thanks to this assumption, we obtain easily the following result.

Corollary 1.2.45. The space $W(V, V^*)$ is a Hilbert space.

We now show that Assumption 1.2.44 holds under certain conditions. See also the remark following the proof of the theorem. The following theorem is abstract and the intuition may not be clear so the reader is referred to the discussion near the end of §2.4.1 for an example.

Theorem 1.2.46. Suppose that

$$u \in \mathcal{W}(V_0, V_0^*) \quad \text{if and only if} \quad T_{(\cdot)}u(\cdot) \in \mathcal{W}(V_0, V_0^*) \tag{T1}$$

and that there exist operators

$$\hat{S}(t) \colon V_0^* \to V_0^* \qquad \text{and} \qquad \hat{D}(t) \colon V_0 \to V_0^*$$

such that for $u \in \mathcal{W}(V_0, V_0^*)$,

$$(T_t u(t))' = \hat{S}(t)u'(t) + \hat{\Lambda}(t)u(t) + \hat{D}(t)u(t) \tag{T2}$$

and

$$\hat{S}(\cdot)u'(\cdot) \in L^2(0,T;V_0^*) \qquad \text{and} \qquad \hat{D}(\cdot)u(\cdot) \in L^2(0,T;V_0^*).$$

Suppose also that $\hat{S}(t)$ and $\hat{D}(t)$ are bounded independently of $t \in [0,T]$, and that $\hat{S}(t)$ has an inverse $\hat{S}(t)^{-1}\colon V_0^* \to V_0^*$ which also is bounded independently of $t \in [0,T]$. Then $W(V,V^*)$ is equivalent to $\mathcal{W}(V_0,V_0^*)$ in the sense of Definition 1.2.44.

Proof. First, suppose $u \in \mathcal{W}(V_0, V_0^*)$. Clearly $\phi_{(\cdot)}u(\cdot) \in L^2_V$ and we need only to show that $\partial^\bullet(\phi_{(\cdot)}u(\cdot)) \in L^2_{V^*}$ exists. Let $\eta \in \mathcal{D}_V(0,T)$ and consider

$$\begin{aligned}
\int_0^T (\phi_t u(t), \dot{\eta}(t))_{H(t)} &= \int_0^T (T_t u(t), (\phi_{-t}\eta(t))')_{H_0} \\
&\qquad\qquad \text{(rewriting the integrand using } \hat{b}(t;\cdot,\cdot) \text{ and (1.15))} \\
&= -\int_0^T \langle \hat{S}(t)u'(t) + \hat{\Lambda}(t)u(t) + \hat{D}(t)u(t), \phi_{-t}\eta(t)\rangle_{V_0^*,V_0} \\
&\qquad\qquad \text{(by (T1) and (T2))} \\
&= -\int_0^T \langle \phi_{-t}^*(\hat{S}(t)u'(t) + \hat{D}(t)u(t)), \eta(t)\rangle_{V^*(t),V(t)} \\
&\quad - \int_0^T \lambda(t;\phi_t u(t), \eta(t)).
\end{aligned} \tag{1.21}$$

This shows that $\partial^\bullet(\phi_{(\cdot)}u(\cdot))$ exists.

Conversely, let $u \in W(V,V^*)$. We need to show the existence of $(\phi_{-(\cdot)}u(\cdot))'$ in $L^2(0,T;V_0^*)$. We start with the weak material derivative condition:

$$\int_0^T \langle \dot{u}(t), \eta(t)\rangle_{V^*(t),V(t)} = -\int_0^T (u(t),\dot{\eta}(t))_{H(t)} - \int_0^T \lambda(t;u(t),\eta(t))$$

for test functions $\eta \in \mathcal{D}_V(0,T)$. Pulling back leads to

$$\begin{aligned}
\int_0^T \langle \phi_t^* \dot{u}(t), \phi_{-t}\eta(t)\rangle_{V_0^*,V_0} &= -\int_0^T \hat{b}(t;\phi_{-t}u(t), (\phi_{-t}\eta(t))') \\
&\quad + \int_0^T \hat{\lambda}(t;\phi_{-t}u(t),\phi_{-t}\eta(t)).
\end{aligned}$$

Using (1.15) and (1.20) and rearranging:

$$\int_0^T (T_t\phi_{-t}u(t), (\phi_{-t}\eta(t))')_{H_0} = -\int_0^T \langle \phi_t^*\dot{u}(t) + \hat{\Lambda}(t)\phi_{-t}u(t), \phi_{-t}\eta(t)\rangle_{V_0^*,V_0}. \quad (1.22)$$

It follows that $T_{(\cdot)}\phi_{-(\cdot)}u(\cdot)$ has a weak derivative, and hence by (T1) as does $\phi_{-(\cdot)}u(\cdot)$. This proves the bijection between $\mathcal{W}(V_0, V_0^*)$ and $W(V, V^*)$.

For the equivalence of norms, let $u \in W(V, V^*)$. From (1.21), we see that

$$\dot{u}(t) = \phi_{-t}^*(\hat{S}(t)(\phi_{-t}u(t))' + \hat{D}(t)\phi_{-t}u(t))$$

which we can bound thanks to the boundedness of $\hat{S}(t)$ and $\hat{D}(t)$:

$$\|\dot{u}(t)\|_{V(t)} \leq C\left(\left\|(\phi_{-t}u(t))'\right\|_{V_0^*} + \|\phi_{-t}u(t)\|_{V_0}\right).$$

So we have achieved $\|u\|_{W(V,V^*)} \leq C_2\left\|\phi_{-(\cdot)}u(\cdot)\right\|_{\mathcal{W}(V_0,V_0^*)}$. For the reverse inequality, we use (T2) and (1.22) to find

$$(\phi_{-t}u(t))' = \hat{S}(t)^{-1}(\phi_t^*\dot{u}(t) - \hat{D}(t)\phi_{-t}u(t)).$$

From this we obtain a bound of the form

$$\left\|(\phi_{-t}u(t))'\right\|_{V_0^*} \leq C\left(\|\dot{u}(t)\|_{V^*(t)} + \|u(t)\|_{V(t)}\right)$$

which implies the result. $\square$

Remark 1.2.47. If we knew that $T_t v_0 \in V_0$ for every $v_0 \in V_0$, then the assumption (T2) would follow from (T1) with $\langle \hat{S}(t)f, v\rangle_{V_0^*,V_0} := \langle f, T_t v\rangle_{V_0^*,V_0}$ and $\hat{D}(t) \equiv 0$.

We are able to specify initial conditions of solutions to PDEs via the following lemma, which is an easy consequence of the continuity of the embedding $\mathcal{W}(V_0, V_0^*) \subset C^0([0,T]; H_0)$.

Lemma 1.2.48. The embedding $W(V, V^*) \subset C_H^0$ holds, hence for any $u \in W(V, V^*)$ the evaluation $t \mapsto u(t)$ is well-defined for every $t \in [0,T]$. Furthermore, we have the inequality

$$\max_{t\in[0,T]} \|u(t)\|_{H(t)} \leq C\|u\|_{W(V,V^*)} \qquad \forall u \in W(V, V^*).$$

This lemma allows us to define the subspace

$$W_0(V, V^*) = \{u \in W(V, V^*) \mid u(0) = 0\}.$$

Definition 1.2.49 (The space $W(V,H)$)**.** Define the space

$$W(V,H) = \{u \in L^2_V \mid \dot{u} \in L^2_H\}.$$

In order to obtain a regularity result, we need to make the following natural assumption, which will also tell us that $W(V,H)$ is a Hilbert space.

Assumption 1.2.50. We assume that there is an evolving space equivalence between $W(V,H)$ and $\mathcal{W}(V_0,H_0)$.

Let us note that this assumption follows if, for example, the assumption (T1) is changed in the natural way and the maps $\hat{S}(t)$ and $\hat{D}(t)$ of Theorem 1.2.46 satisfy $\hat{S}(t)\colon H_0 \to H_0$ and $\hat{D}(t)\colon V_0 \to H_0$, with both maps and $\hat{S}(t)^{-1}$ being bounded independently of $t \in [0,T]$, and if $\hat{S}(\cdot)u'(\cdot)$, $\hat{D}(\cdot)u(\cdot) \in L^2(0,T;H_0)$ for $u \in \mathcal{W}(V_0,H_0)$.

Some density results With the help of the density result in Lemma 1.2.1, it is easy to prove the following lemma.

Lemma 1.2.51. The space $\mathcal{D}_V[0,T]$ in dense in $W(V,V^*)$.

The next few results are necessary to prove Lemma 1.3.5, which turns out to be vital for one of our existence proofs.

Lemma 1.2.52. For every $\eta \in \mathcal{D}_V(0,T)$, there exists a sequence $\{\eta_n\} \subset \mathcal{D}_V(0,T)$ of the form

$$\eta_n(t) = \sum_{j=1}^{n} \zeta_j(t)\phi_t w_j \qquad \text{where } \zeta_j \in \mathcal{D}(0,T) \text{ and } w_j \in V_0,$$

such that $\eta_n \to \eta$ in $W(V,V^*)$.

Proof. It suffices to show that for every $\psi \in \mathcal{D}((0,T);V_0)$, there exists a sequence $\{\psi_n\} \subset \mathcal{D}((0,T);V_0)$ of the form

$$\psi_n(t) = \sum_{j=1}^{n} \zeta_j(t) w_j \qquad \text{where } \zeta_j \in \mathcal{D}(0,T) \text{ and } w_j \in V_0,$$

such that $\psi_n \to \psi$ in $\mathcal{W}(V_0,V_0^*)$.

Let w_j be an orthonormal basis for V_0. Given $\psi \in \mathcal{D}((0,T);V_0)$, define

$$\psi_n(t) = \sum_{j=1}^{n} (\psi(t), w_j)_{V_0} w_j,$$

i.e., $\zeta_j(t) = (\psi(t), w_j)_{V_0}$. It is clear that ζ_j vanishes at the boundary (since ψ does), and $\zeta_j^{(m)}(t) = (\psi^{(m)}(t), w_j)_{V_0}$ also implies that $\zeta_j \in \mathcal{D}(0,T)$. What remains to be checked is that $\psi_n \to \psi$ in $\mathcal{W}(V_0, V_0^*)$. We have the pointwise convergence $\psi_n(t) \to \psi(t)$ in V_0 because w_j is a basis, and there is also the uniform bound $\|\psi_n(t)\|_{V_0} \leq \|\psi(t)\|_{V_0}$. So by the dominated convergence theorem,

$$\psi_n \to \psi \qquad \text{in } L^2(0,T;V_0).$$

The same reasoning applied to ψ_n' allows us to conclude. $\square$

Transport theorem Like in part (3) of Lemma 1.2.1, we want to differentiate the inner product on $H(t)$. Writing Lemma 1.2.39 in different notation, we obtain for u, $v \in C_H^1$ the transport theorem for C_H^1 functions:

$$\frac{d}{dt}(u(t), v(t))_{H(t)} = (\dot{u}(t), v(t))_{H(t)} + (u(t), \dot{v}(t))_{H(t)} + \lambda(t; u(t), v(t)).$$

We can obtain a formula for general functions u, $v \in W(V, V^*)$ by means of a density argument.

Theorem 1.2.53 (Transport theorem)**.** For all u, $v \in W(V, V^*)$, the map

$$t \mapsto (u(t), v(t))_{H(t)}$$

is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}(u(t), v(t))_{H(t)} = \langle \dot{u}(t), v(t)\rangle_{V^*(t),V(t)} + \langle \dot{v}(t), u(t)\rangle_{V^*(t),V(t)} + \lambda(t; u(t), v(t))$$

for almost every $t \in [0,T]$.

Proof. Given $u \in W(V, V^*)$, by Lemma 1.2.51, there exists a sequence $u_m \in \mathcal{D}_V[0,T]$ converging to u in $W(V, V^*)$. By the transport theorem for C_H^1 functions, the u_m satisfy

$$\frac{d}{dt}\|u_m(t)\|_{H(t)}^2 = 2(\dot{u}_m(t), u_m(t))_{H(t)} + \lambda(t; u_m(t), u_m(t)).$$

This statement written in terms of weak derivatives is that for any $\zeta \in \mathcal{D}(0,T)$, it holds that

$$-\int_0^T \|u_m(t)\|_{H(t)}^2 \zeta'(t) = \int_0^T \left(2\langle \dot{u}_m(t), u_m(t)\rangle_{V^*(t),V(t)} + \lambda(t; u_m(t), u_m(t))\right)\zeta(t). \tag{1.23}$$

Now we must pass to the limit in this equation. For the left hand side, because $u_m \to u$ in L^2_H, we have by the reverse triangle inequality

$$\int_0^T \left| \, \|u_m(t)\|_{H(t)} - \|u(t)\|_{H(t)} \, \right|^2 \leq \int_0^T \|u_m(t) - u(t)\|^2_{H(t)} \to 0,$$

i.e., $\|u_m(\cdot)\|_{H(\cdot)} \to \|u(\cdot)\|_{H(\cdot)}$ in $L^2(0,T)$, which implies that

$$\|u_m(\cdot)\|^2_{H(\cdot)} \to \|u(\cdot)\|^2_{H(\cdot)} \qquad \text{in } L^1(0,T).$$

Clearly, the functional $F \colon L^1(0,T) \to \mathbb{R}$, defined

$$F(y) = \int_0^T y(t)\zeta'(t),$$

is an element of $L^1(0,T)^*$ because $\zeta'(t)$ is bounded. Therefore, we have convergence of the left hand side of (1.23):

$$-\int_0^T \|u_m(t)\|^2_{H(t)} \, \zeta'(t) \to -\int_0^T \|u(t)\|^2_{H(t)} \, \zeta'(t).$$

To deal with the terms on the right hand side of (1.23), we require the estimates

$$\begin{aligned}
&|\langle \dot{u}_m(t), u_m(t)\rangle_{V^*(t),V(t)} - \langle \dot{u}(t), u(t)\rangle_{V^*(t),V(t)}| \\
&\leq \|\dot{u}_m(t)\|_{V^*(t)} \, \|u_m(t) - u(t)\|_{V(t)} + \|\dot{u}_m(t) - \dot{u}(t)\|_{V^*(t)} \, \|u(t)\|_{V(t)}
\end{aligned}$$

and

$$\begin{aligned}
&|\lambda(t; u_m(t), u_m(t)) - \lambda(t; u(t), u(t))| \\
&\quad \leq C_1 \left(\|u_m(t)\|_{H(t)} \, \|u_m(t) - u(t)\|_{H(t)} + \|u_m(t) - u(t)\|_{H(t)} \, \|u(t)\|_{H(t)} \right).
\end{aligned}$$

With these, it is easy to show that

$$\begin{aligned}
\Bigg| \int_0^T &\left(2\langle \dot{u}_m(t), u_m(t)\rangle_{V^*(t),V(t)} + \lambda(t; u_m(t), u_m(t)) \right) \zeta(t) \\
&- \int_0^T \left(2\langle \dot{u}(t), u(t)\rangle_{V^*(t),V(t)} + \lambda(t; u(t), u(t)) \right) \zeta(t) \Bigg| \to 0.
\end{aligned}$$

In other words, as $m \to \infty$, the equation (1.23) becomes

$$-\int_0^T \|u(t)\|^2_{H(t)} \, \zeta'(t) = \int_0^T \left(2\langle \dot{u}(t), u(t)\rangle_{V^*(t),V(t)} + \lambda(t; u(t), u(t)) \right) \zeta(t), \tag{1.24}$$

which is precisely the statement

$$\frac{d}{dt}\|u(t)\|_{H(t)}^2 = 2\langle \dot{u}(t), u(t)\rangle_{V^*(t),V(t)} + \lambda(t; u(t), u(t))$$

in the sense of distributions. From this, it follows by the polarisation identity that

$$\frac{d}{dt}(u(t), v(t))_{H(t)} = \langle \dot{u}(t), v(t)\rangle_{V^*(t),V(t)} + \langle \dot{v}(t), u(t)\rangle_{V^*(t),V(t)} + \lambda(t; u(t), v(t)) \tag{1.25}$$

holds in the weak sense. So we have shown the transport theorem in the weak sense. However, because the right hand side of the above is in $L^1(0,T)$ (since the right hand side of (1.24) holds for every $\zeta \in \mathcal{D}(0,T)$) and because $(u(t), v(t))_{H(t)} \in L^1(0,T)$, it follows that $(u(t), v(t))_{H(t)}$ is a.e. equal to an absolutely continuous function, with (classical) derivative a.e., and therefore (1.25) exists in the classical sense. □

We shall use the following corollary frequently without referencing in future sections.

Corollary 1.2.54 (Integration by parts)**.** For all u, $v \in W(V, V^*)$, the integration by parts formula

$$\begin{aligned}(u(T), v(T))_{H(T)} &- (u(0), v(0))_{H_0} \\ &= \int_0^T \langle \dot{u}(t), v(t)\rangle_{V^*(t),V(t)} + \langle \dot{v}(t), u(t)\rangle_{V^*(t),V(t)} + \lambda(t; u(t), v(t))\,\mathrm{d}t\end{aligned}$$

holds.

1.3 Precise formulation of PDE on abstract evolving Hilbert space

Having built up the essential function spaces and results, we are now in a position to formulate PDEs on evolving spaces. We continue with the framework and notation of the previous sections; we reiterate in particular Assumptions 1.2.27, 1.2.35, and 1.2.44 (which relate respectively to the compatibility of the evolving Hilbert spaces, a well-defined material derivative, and the evolving space equivalence). We are interested in the existence and uniqueness of solutions $u \in W(V, V^*)$ to equations of the form

$$\begin{aligned} L\dot{u} + Au + \Lambda u &= f \quad \text{in } L^2_{V^*} \\ u(0) &= u_0 \quad \text{in } H_0, \end{aligned} \tag{$\mathbf{P}$}$$

where we identify

$$\begin{aligned}(L\dot{u})(t) &= L(t)\dot{u}(t)\\ (Au)(t) &= A(t)u(t)\\ (\Lambda u)(t) &= \Lambda(t)u(t),\end{aligned}$$

with $L(t)$ and $A(t)$ being linear operators that satisfy the minimal assumptions given below, and

$$\Lambda(t)\colon H(t) \to H^*(t) \quad \text{is defined by} \quad \langle \Lambda(t)v, w\rangle_{H^*(t),H(t)} := \lambda(t; v, w),$$

with $\lambda(t;\cdot,\cdot)$ the bilinear form in the definition of the weak material derivative (Definition 1.2.36). Note that $\Lambda(t)$ is symmetric in the sense that $\langle \Lambda(t)v, w\rangle_{H^*(t),H(t)} = \langle \Lambda(t)w, v\rangle_{H^*(t),H(t)}$.

Remark 1.3.1. We showed in Lemma 1.2.48 that specifying the initial condition as in (**P**) is well-defined.

Assumptions 1.3.2 (Assumptions on $L(t)$)**.** In the following, all constants C_i are positive and independent of $t \in [0, T]$. We shall assume that for all $g \in L^2_{V^*}$,

$$Lg \in L^2_{V^*} \qquad \text{and} \qquad C_1 \|g\|_{L^2_{V^*}} \leq \|Lg\|_{L^2_{V^*}} \leq C_2 \|g\|_{L^2_{V^*}}. \tag{L1}$$

We suppose that the restriction $L|_{L^2_H}$ satisfies

$$L|_{L^2_H} \colon L^2_H \to L^2_H,$$

we identify $(L|_{L^2_H} h)(t) =: L_H(t)h(t)$, and we suppose that

$$\begin{aligned}&L_H(t)\colon H(t) \to H(t) \text{ is symmetric, and}\\ &L_H(t)\colon V(t) \to V(t).\end{aligned}$$

We simply write L and $L(t)$ for the above restrictions. Furthermore, for almost every $t \in [0, T]$, we assume

$$\langle L(t)g, v\rangle_{V^*(t),V(t)} = \langle g, L(t)v\rangle_{V^*(t),V(t)} \qquad \forall g \in V^*(t),\ \forall v \in V(t) \tag{L2}$$

$$\|L(t)h\|_{H(t)} \leq C_3 \|h\|_{H(t)} \qquad \forall h \in H(t) \tag{L3}$$

$$(L(t)h, h)_{H(t)} \geq C_4 \|h\|^2_{H(t)} \qquad \forall h \in H(t) \tag{L4}$$

$$Lv \in L^2_V \qquad \forall v \in L^2_V \tag{L5}$$

$$v \in W(V, V^*) \iff Lv \in W(V, V^*), \tag{L6}$$

and we suppose the existence of a (linear symmetric) map $\dot{L}\colon L^2_V \to L^2_{V^*}$ (and we identify $(\dot{L}v)(t) =: \dot{L}(t)v(t)$) satisfying

$$\partial^\bullet(Lv) = \dot{L}v + L\dot{v} \in L^2_{V^*} \qquad \forall v \in W(V, V^*) \tag{L7}$$

$$\|\dot{L}(t)v\|_{V^*(t)} \leq C_5 \|v\|_{H(t)} \qquad \forall v \in V(t). \tag{L8}$$

Assumptions 1.3.3 (Assumptions on $A(t)$)**.** Suppose that the map

$$t \mapsto \langle A(t)v(t), w(t)\rangle_{V^*(t),V(t)} \qquad \forall v, w \in L^2_V$$

is measurable, and that there exist positive constants C_1, C_2 and C_3 independent of t such that the following holds for almost every $t \in [0, T]$:

$$\langle A(t)v, v\rangle_{V^*(t),V(t)} \geq C_1 \|v\|^2_{V(t)} - C_2 \|v\|^2_{H(t)} \qquad \forall v \in V(t) \tag{A1}$$

$$|\langle A(t)v, w\rangle_{V^*(t),V(t)}| \leq C_3 \|v\|_{V(t)} \|w\|_{V(t)} \qquad \forall v, w \in V(t). \tag{A2}$$

Observe that we have generalised the PDE (1.2) by introducing the operator L. The standard equation

$$\dot{u} + Au + \Lambda u = f$$

is a special case of (**P**) when $L = \text{Id}$. Our demands in Assumptions 1.3.2 are (of course) automatically met in this case. Also, there is no loss of generality by considering the equation (**P**) instead of the more natural equation $L\dot{u} + Au = f$. We include the operator Λ purely because it is convenient in applications (such as those in Chapter 2).

Implicit in (**P**) is the claim that Au and Λu are elements of $L^2_{V^*}$. The fact $Au \in L^2_{V^*}$ follows by the weak (and thus strong) measurability of $t \mapsto \phi_t^* A(t)u(t)$ and the boundedness of $A(t)$, and similarly one obtains the result $\Lambda u \in L^2_{V^*}$. Let us mention an important consequence of the transport theorem (Theorem 1.2.53) and assumptions (L2), (L6) and (L7).

Lemma 1.3.4. For every v, $w \in W(V, V^*)$, the map $t \mapsto (L(t)v(t), w(t))_{H(t)}$ is absolutely continuous with derivative

$$\begin{aligned}\frac{d}{dt}(L(t)v(t), w(t))_{H(t)} &= \langle L(t)\dot{v}(t), w(t)\rangle_{V^*(t),V(t)} + \langle L(t)\dot{w}(t), v(t)\rangle_{V^*(t),V(t)} \\ &\quad + \langle M(t)v(t), w(t)\rangle_{V^*(t),V(t)}\end{aligned} \tag{1.26}$$

almost everywhere, where $M(t)\colon V(t) \to V^*(t)$ is the operator

$$\langle M(t)v, w\rangle_{V^*(t),V(t)} := \langle \dot{L}(t)v, w\rangle_{V^*(t),V(t)} + \langle \Lambda(t)L(t)v, w\rangle_{V^*(t),V(t)}$$

which generates the bounded bilinear form $m(t;\cdot,\cdot)\colon V(t) \times V(t) \to \mathbb{R}$:

$$m(t; v, w) := \langle M(t)v, w\rangle_{V^*(t),V(t)}.$$

To conclude this preliminary subsection we state and prove the following lemma which is used in §1.6.4.

Lemma 1.3.5. Let $u \in L^2_V$ and $g \in L^2_{V^*}$. Then

$$\dot{u} \in L^2_{V^*} \text{ exists and } L\dot{u} = g$$

if and only if

$$\frac{d}{dt}(L(t)u(t), \phi_t v_0)_{H(t)} = \langle g(t) + M(t)u(t), \phi_t v_0\rangle_{V^*(t),V(t)} \quad \text{for all} \quad v_0 \in V_0 \qquad (1.27)$$

in the weak sense.

Proof of Lemma 1.3.5. If $u \in W(V, V^*)$ and $L\dot{u} = g$, then (1.27) follows easily by utilising $\partial^\bullet(\phi_t v_0) = 0$ and the previous lemma. For the converse, first, we see from Lemma 1.2.52 that given any $\eta \in \mathcal{D}_V(0,T)$, there exist functions $\eta_n \in \mathcal{D}_V(0,T)$ of the form

$$\eta_n(t) = \sum_j \zeta_j(t)\phi_t w_j$$

with $\zeta_j \in \mathcal{D}(0,T)$ and $w_j \in V_0$ such that $\|\eta - \eta_n\|_{W(V,V^*)} \to 0$. Now, (1.27) states that

$$\int_0^T (L(t)u(t), \zeta'(t)\phi_t v_0)_{H(t)} = -\int_0^T \langle g(t) + M(t)u(t), \zeta(t)\phi_t v_0\rangle_{V^*(t),V(t)}$$

holds for all $\zeta \in \mathcal{D}(0,T)$ and all $v_0 \in V_0$. In particular, we may pick $\zeta = \zeta_j$ and $v_0 = w_j$ and sum up over j to obtain

$$\int_0^T (L(t)u(t), \dot{\eta}_n(t))_{H(t)} = -\int_0^T \langle g(t) + M(t)u(t), \eta_n(t)\rangle_{V^*(t),V(t)}.$$

Passing to the limit and using the convergence above, we find

$$\begin{aligned}\int_0^T (L(t)u(t), \dot{\eta}(t))_{H(t)} &= -\int_0^T \langle g(t) + M(t)u(t), \eta(t)\rangle_{V^*(t),V(t)} \\ &= -\int_0^T \langle g(t) + \dot{L}(t)u(t) + \Lambda(t)L(t)u(t), \eta(t)\rangle_{V^*(t),V(t)}\end{aligned}$$

for arbitrary $\eta \in \mathcal{D}_V(0,T)$, i.e., we have the existence of $\partial^\bullet(Lu) = g + \dot{L}u \in L^2_{V^*}$ which, thanks to assumptions (L6) and (L7) implies that $L\dot{u} = g$. □

1.4 Well-posedness and regularity theorems

We begin with a well-posedness theorem which is proved in §1.5. A sketch of a second proof will be presented in §1.6.4 where we utilise a Galerkin method.

Theorem 1.4.1 (Well-posedness of (**P**))**.** Under the assumptions in Assumptions 1.3.2 and 1.3.3, for $f \in L^2_{V^*}$ and $u_0 \in H_0$, there is a unique solution $u \in W(V,V^*)$ satisfying (**P**) such that

$$\|u\|_{W(V,V^*)} \leq C\left(\|u_0\|_{H_0} + \|f\|_{L^2_{V^*}}\right).$$

Now, suppose we now know that $f \in L^2_H$ and $u_0 \in V_0$. Can we expect the same regularity on the solution u as holds in the case of stationary spaces? It turns out that we can obtain $\dot{u} \in L^2_H$ under some additional assumptions, including some on the differentiability of $A(t)$.

Before we list these assumptions, let us just note that if we define bilinear forms $l(t;\cdot,\cdot)\colon V^*(t) \times V(t) \to \mathbb{R}$ and $a(t;\cdot,\cdot)\colon V(t) \times V(t) \to \mathbb{R}$ to satisfy

$$\begin{aligned}l(t;g,w) &:= \langle L(t)g, w\rangle_{V^*(t),V(t)} \\ a(t;v,w) &:= \langle A(t)v, w\rangle_{V^*(t),V(t)},\end{aligned}$$

then the problem (**P**) is in fact equivalent to

$$\begin{aligned}l(t;\dot{u}(t),v) + a(t;u(t),v) + \lambda(t;u(t),v) &= \langle f(t), v\rangle_{V^*(t),V(t)} \\ u(0) &= u_0\end{aligned} \tag{1.28}$$

for all $v \in V(t)$ and for almost every $t \in [0,T]$ (the null set is independent of v).

Similarly, if $f \in L^2_H$ and $\dot{u} \in L^2_H$, then (**P**) is equivalent to

$$\begin{aligned} l(t;\dot{u}(t),v) + a(t;u(t),v) + \lambda(t;u(t),v) &= (f(t),v)_{H(t)} \\ u(0) &= u_0 \end{aligned} \tag{\textbf{P'}}$$

for all $v \in V(t)$ and for almost every $t \in [0,T]$, where now $l(t;\cdot,\cdot)\colon H(t) \times H(t) \to \mathbb{R}$ is $l(t;\cdot,\cdot) = (L(t)\cdot,\cdot)_{H(t)}$. It is this form of the problem that turns out to be more convenient to work with to show regularity. To see the equivalence, for one side, we may take the duality pairing of (**P**) with $v = \xi\phi_{(\cdot)}v_0$ where $v_0 \in V_0$ and $\xi \in \mathcal{D}(0,T)$; then an argument involving the separability of V_0 gives (**P'**). The converse follows by the density of simple measurable functions in L^2_V (see Lemma 1.2.12).

Since V_0 is separable, we may find a basis $\{\chi^0_j\}$, by which we mean that for all $N \in \mathbb{N}$, the set $\{\chi^0_j\}_{j=1}^N$ is linearly independent and finite linear combinations of χ^0_j are dense in V_0.

Assumption 1.4.2. We assume that there exists a basis $\{\chi^0_j\}_{j\in\mathbb{N}}$ of V_0 and a sequence $\{u_{0N}\}_{N\in\mathbb{N}}$ with $u_{0N} \in \text{span}\{\chi^0_1, ..., \chi^0_N\}$ for each N, such that

$$u_{0N} \to u_0 \qquad \text{in } V_0 \tag{B1}$$

$$\|u_{0N}\|_{H_0} \le C_1 \|u_0\|_{H_0} \tag{B2}$$

$$\|u_{0N}\|_{V_0} \le C_2 \|u_0\|_{V_0} \tag{B3}$$

where C_1 and C_2 do not depend on N or u_0.

Remark 1.4.3. Such a basis as required by the last assumption always exists if $V_0 \subset H_0$ is compact thanks to Hilbert–Schmidt theory. In fact, in such a case we can find a basis χ^0_j which is orthonormal in H_0 and orthogonal in V_0. In the context of Sobolev spaces on domains, we know that in general $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is not compact if Ω is unbounded [27, Remark 14, §9.3], so it is not clear that the previous assumption holds in which case this argument does not give regularity of the solution. However, there are certainly other methods one could use to prove time regularity of the solution.

Let $AC([0,T])$ be the space of absolutely continuous functions from $[0,T]$ into $\mathbb{R}$.

Definition 1.4.4. We define $\chi^t_j := \phi_t(\chi^0_j)$ and the space

$$\tilde{C}^1_V := \{u \mid u(t) = \sum_{j=1}^m \alpha_j(t)\chi^t_j,\ m \in \mathbb{N},\ \alpha_j \in AC([0,T]) \text{ and } \alpha'_j \in L^2(0,T)\}.$$

Note that $\tilde{C}^1_V \subset C^0_V$ and $\tilde{C}^1_V \subset W(V,V)$.

Remark 1.4.5. Note that if $u \in \tilde{C}^1_V$ with $u(t) = \sum_{j=1}^m \alpha_j(t)\chi_j^t$ as in the definition then $\dot{u}(t) = \sum_{j=1}^m \alpha_j'(t)\chi_j^t$. We skip the proof which is straightforward: just use the definition of the weak material derivative and perform some manipulations. We could not have calculated the strong material derivative of u via the formula (1.14) because the pullback

$$\phi_{-(\cdot)}u(\cdot) = \sum_{j=1}^n \alpha_j(\cdot)\chi_j^0$$

is not necessarily in $C^1([0,T];V_0)$ since the α_j are not necessarily C^1.

Assumptions 1.4.6 (Further assumptions on $a(t;\cdot,\cdot)$)**.** Suppose that $a(t;\cdot,\cdot)$ has the form

$$a(t;\cdot,\cdot) = a_s(t;\cdot,\cdot) + a_n(t;\cdot,\cdot)$$

where

$$\begin{aligned}
a_s(t;\cdot,\cdot)&\colon V(t)\times V(t) \to \mathbb{R}\\
a_n(t;\cdot,\cdot)&\colon V(t)\times H(t) \to \mathbb{R}
\end{aligned}$$

are bilinear forms (we allow the possibility $a_n \equiv 0$) such that the map

$$t \mapsto a_s(t;y(t),y(t)) \text{ is absolutely continuous on } [0,T] \text{ for all } y \in \tilde{C}^1_V. \tag{A3}$$

Suppose also that there exist positive constants C_1, C_2 and C_3 independent of t such that for almost every $t \in [0,T]$,

$$\begin{aligned}
|a_n(t;v,w)| &\leq C_1 \|v\|_{V(t)} \|w\|_{H(t)} && \forall v \in V(t), w \in H(t) && \text{(A4)}\\
|a_s(t;v,w)| &\leq C_2 \|v\|_{V(t)} \|w\|_{V(t)} && \forall v,w \in V(t) && \text{(A5)}\\
a_s(t;v,v) &\geq 0 && \forall v \in V(t) && \text{(A6)}\\
\frac{d}{dt}a_s(t;y(t),y(t)) &= 2a_s(t;y(t),\dot{y}(t)) + r(t;y(t)) && \forall y \in \tilde{C}^1_V, && \text{(A7)}
\end{aligned}$$

where the $\frac{d}{dt}$ here is the classical derivative, and $r(t;\cdot)\colon V(t) \to \mathbb{R}$ satisfies

$$|r(t;v)| \leq C_3 \|v\|^2_{V(t)} \qquad \forall v \in V(t). \tag{A8}$$

Remark 1.4.7. Note that we require only one part of the bilinear form $a(t;\cdot,\cdot)$ to be differentiable; however, any potentially non-differentiable terms require the stronger boundedness condition (A4). This weakening of the standard differentiability assumption is useful in §2.5.2.

As alluded to above, it is permissible to take $a_n \equiv 0$ so that $a \equiv a_s$. In this case, we are in the same situation as in Assumptions 1.3.3 except with the addition of (A3), (A6), (A7), and (A8).

We have the following regularity result proved in §1.6.

Theorem 1.4.8 (Regularity of the solution to (**P**))**.** Under the assumptions in Assumptions 1.3.2, 1.3.3, 1.4.2, and 1.4.6, if $f \in L^2_H$ and $u_0 \in V_0$, the unique solution u of (**P**) from Theorem 1.4.1 satisfies the regularity $u \in W(V,H)$ and the estimate

$$\|u\|_{W(V,H)} \leq C \left(\|u_0\|_{V_0} + \|f\|_{L^2_H} \right).$$

1.5 Proof of well-posedness

We use a generalisation of the Lax–Milgram theorem sometimes called the Banach–Nečas–Babuška theorem [61, §2.1.3] to establish existence.

Theorem 1.5.1 (Banach–Nečas–Babuška)**.** Let X be a Banach space and let Y be a reflexive Banach space. Suppose $d(\cdot,\cdot)\colon X \times Y \to \mathbb{R}$ is a bounded bilinear form and $f \in Y^*$. Then there is a unique solution $x \in X$ to the problem

$$d(x,y) = \langle f, y\rangle_{Y^*,Y} \qquad \text{for all } y \in Y$$

satisfying

$$\|x\|_X \leq C\,\|f\|_{Y^*} \tag{1.29}$$

if and only if

1. There exists $\alpha > 0$ such that

$$\inf_{x \in X} \sup_{y \in Y} \frac{d(x,y)}{\|x\|_X \|y\|_Y} \geq \alpha. \qquad \text{(“inf-sup condition”)}$$

2. For arbitrary $y \in Y$, if

$$d(x,y) = 0 \text{ holds for all } x \in X,$$

then $y = 0$.

Moreover, the estimate (1.29) holds with the constant $C = \frac{1}{\alpha}$.

Remark 1.5.2 (Relation to the Lax–Milgram lemma)**.** Above, if we pick $X = Y$ a Hilbert space, then the previous theorem implies the Lax–Milgram lemma. Indeed, let us assume that the conditions of Lax–Milgram are satisfied by d. By coercivity,

$$\sup_{y \in X} \frac{d(x,y)}{\|y\|_X} \geq \frac{d(x,x)}{\|x\|_X} \geq C \|x\|_X$$

which implies the inf-sup condition. Secondly, for arbitrary y,

$$\sup_{x \in X} d(x,y) \geq d(y,y) \geq C \|y\|_X^2 ,$$

thus if the left hand side is zero, y is also zero.

Remark 1.5.3 (Other approaches)**.** The standard Lax–Milgram approach would not work for the well-posedness since the bilinear form $b\colon W(V,V^*) \times W(V,V^*) \to \mathbb{R}$ defined by $b(u,v) := \langle L\dot{u} + Au + \Lambda u, v \rangle$ is not coercive in the space $W(V,V^*)$. It is an open problem whether the method of time-discretisation can be adapted to moving spaces (rather than pulling back onto a reference domain). The difficulty lies in interpolation and obtaining estimates to pass to the limit in the discretisation parameter.

Recall the equation (**P**):

$$\begin{aligned} L\dot{u} + Au + \Lambda u &= f \qquad \text{in } L^2_{V^*} \\ u(0) &= u_0 \end{aligned}$$

where $f \in L^2_{V^*}$ and $u_0 \in H_0$. By considering a suitable initial value problem on a fixed domain we know that there is a function $y \in \mathcal{W}(V_0, V_0^*)$ with $y(0) = u_0$ and

$$\|y\|_{\mathcal{W}(V_0,V_0^*)} \leq C \|u_0\|_{H_0} .$$

Then the function $\tilde{y}(\cdot) = \phi_{(\cdot)} y(\cdot)$ is such that $\tilde{y} \in W(V,V^*)$ with $\tilde{y}(0) = u_0$. So then we can transform (**P**) into a PDE with zero initial condition if we set $w = u - \tilde{y}$:

$$\begin{aligned} L\dot{w} + Aw + \Lambda w &= \tilde{f} \\ w(0) &= 0 \end{aligned} \tag{$\mathbf{P_0}$}$$

where $\tilde{f} := f - L\partial^\bullet \tilde{y} - A\tilde{y} - \Lambda \tilde{y} \in L^2_{V^*}$. It is clear that well-posedness of ($\mathbf{P_0}$) translates into well-posedness of (**P**). The idea is to apply Theorem 1.5.1 to the problem ($\mathbf{P_0}$) with $X = W_0(V, V^*)$, $Y = L^2_V$, and the bilinear form

$$d(u,v) = \langle L\dot{u}, v\rangle_{L^2_{V^*},L^2_V} + \langle Au, v\rangle_{L^2_{V^*},L^2_V} + \langle \Lambda u, v\rangle_{L^2_{V^*},L^2_V}.$$

Remark 1.5.4. The space $W_0(V, V^*)$ is indeed a Hilbert space because by Lemma 1.2.48, it is a closed linear subspace of $W(V, V^*)$.

The arguments in the next two lemmas follow §4 in [98]. See also [61, §6.1.2]. The next lemma goes towards proving the inf-sup condition.

Lemma 1.5.5. For all $w \in W_0(V, V^*)$, there exists a function $v_w \in L^2_V$ such that

$$\langle L\dot{w}, v_w\rangle_{L^2_{V^*},L^2_V} + \langle Aw, v_w\rangle_{L^2_{V^*},L^2_V} + \langle \Lambda w, v_w\rangle_{L^2_{V^*},L^2_V} \geq C \|w\|_{W(V,V^*)} \|v_w\|_{L^2_V}.$$

Proof. This proof requires two estimates.

First estimate Let $w \in W_0(V, V^*)$ and set $w_\gamma(t) = e^{-\gamma t}w(t)$. Note that $w_\gamma \in W_0(V, V^*)$ too with $\dot{w}_\gamma(t) = e^{-\gamma t}\dot{w}(t) - \gamma w_\gamma(t)$, so

$$\langle L(t)\dot{w}_\gamma(t), w(t)\rangle_{V^*(t),V(t)} = \langle L(t)\dot{w}(t) - \gamma L(t)w(t), w_\gamma(t)\rangle_{V^*(t),V(t)}.$$

Rearranging, integrating, and then using (1.26):

$$\begin{aligned}\langle L\dot{w}, w_\gamma\rangle_{L^2_{V^*},L^2_V} &= \frac{1}{2}\left(\langle L\dot{w}, w_\gamma\rangle_{L^2_{V^*},L^2_V} + \langle L\dot{w}_\gamma, w\rangle_{L^2_{V^*},L^2_V}\right) + \frac{1}{2}\gamma(Lw, w_\gamma)_{L^2_H} \quad (1.30)\\ &= \frac{1}{2}\int_0^T \frac{d}{dt}(L(t)w(t), w_\gamma(t))_{H(t)} - \frac{1}{2}\langle Mw, w_\gamma\rangle_{L^2_{V^*},L^2_V}\\ &\quad + \frac{1}{2}\gamma(Lw, w_\gamma)_{L^2_H}\\ &\geq -\frac{1}{2}\langle Mw, w_\gamma\rangle_{L^2_{V^*},L^2_V} + \frac{1}{2}\gamma(Lw, w_\gamma)_{L^2_H}\end{aligned}$$

as $(L(T)w(T), w_\gamma(T))_{H(T)} \geq 0$ by (L4). Hence

$$\begin{aligned}&\langle L\dot{w}, w_\gamma\rangle_{L^2_{V^*},L^2_V} + \langle Aw, w_\gamma\rangle_{L^2_{V^*},L^2_V} + \langle \Lambda w, w_\gamma\rangle_{L^2_{V^*},L^2_V}\\ &\geq \langle Aw, w_\gamma\rangle_{L^2_{V^*},L^2_V} + \langle \Lambda w, w_\gamma\rangle_{L^2_{V^*},L^2_V} - \frac{1}{2}\langle Mw, w_\gamma\rangle_{L^2_{V^*},L^2_V} + \frac{1}{2}\gamma(Lw, w_\gamma)_{L^2_H}\\ &\geq \int_0^T e^{-\gamma t}\left(C_1 \|w(t)\|^2_{V(t)} - C_2 \|w(t)\|^2_{H(t)}\right) - \frac{1}{2}\int_0^T C_3 e^{-\gamma t} \|w(t)\|^2_{H(t)}\\ &\quad + \frac{\gamma C_4}{2}\int_0^T e^{-\gamma t} \|w(t)\|^2_{H(t)}\\ &\qquad \text{(by the coercivity of } A(t) \text{ and } L(t) \text{ and the boundedness of } \Lambda(t) \text{ and } M(t))\\ &= C_1 \int_0^T e^{-\gamma t} \|w(t)\|^2_{V(t)} + \frac{\gamma C_4 - C_3 - 2C_2}{2}\int_0^T e^{-\gamma t} \|w(t)\|^2_{H(t)}\end{aligned}$$

Note that we used Young's inequality in conjunction with the boundedness of $M(t)$ above. If we choose γ such that $\gamma C_4 > C_3 + 2C_2$, this implies

$$\langle L\dot{w}, w_\gamma\rangle_{L^2_{V^*},L^2_V} + \langle Aw, w_\gamma\rangle_{L^2_{V^*},L^2_V} + \langle \Lambda w, w_\gamma\rangle_{L^2_{V^*},L^2_V} \geq e^{-\gamma T} C_1 \|w\|^2_{L^2_V}. \tag{E1}$$

Second estimate Now, by the Riesz representation theorem, there exists $z \in L^2_V$ such that

$$\langle L\dot{w}, v\rangle_{L^2_{V^*},L^2_V} = (z, v)_{L^2_V} \qquad \text{for all } v \in L^2_V \tag{1.31}$$

with $\|z\|_{L^2_V} = \|L\dot{w}\|_{L^2_{V^*}}$. We have

$$\begin{aligned}
\langle L\dot{w} + Aw + \Lambda w, z\rangle_{L^2_{V^*},L^2_V} &\geq \|z\|^2_{L^2_V} - C_5 \int_0^T \|w(t)\|_{V(t)} \|z(t)\|_{V(t)} \\
&\qquad\qquad \text{(by (1.31) and the bounds on } A \text{ and } \Lambda) \\
&\geq C_6 \|z\|^2_{L^2_V} - C_7 \|w\|^2_{L^2_V} \qquad \text{(using Young's inequality)} \\
&= C_6 \|L\dot{w}\|^2_{L^2_{V^*}} - C_7 \|w\|^2_{L^2_V}.
\end{aligned} \tag{E2}$$

Combining the estimates Estimate (E2) gives us control of $L\dot{w}$ at the expense of w, but the latter is controlled by estimate (E1). So let us put $v_w := z + \mu w_\gamma$ where $\mu > 0$ is a constant to be determined and consider:

$$\begin{aligned}
&\langle L\dot{w}, v_w\rangle_{L^2_{V^*},L^2_V} + \langle Aw, v_w\rangle_{L^2_{V^*},L^2_V} + \langle \Lambda w, v_w\rangle_{L^2_{V^*},L^2_V} \\
&\qquad \geq C_6 \|L\dot{w}\|^2_{L^2_{V^*}} - C_7 \|w\|^2_{L^2_V} + \mu e^{-\gamma T} C_1 \|w\|^2_{L^2_V} \\
&\qquad \geq C_6 \|L\dot{w}\|^2_{L^2_{V^*}} + C_8 \|w\|^2_{L^2_V} \qquad \text{(if } \mu \text{ is large enough)} \\
&\qquad \geq C_9 \|w\|^2_{W(V,V^*)}
\end{aligned}$$

thanks to (L1). Finally, because

$$\begin{aligned}
\|v_w\|_{L^2_V} &\leq \|z\|_{L^2_V} + \mu \|w_\gamma\|_{L^2_V} \\
&= \|L\dot{w}\|_{L^2_{V^*}} + \mu \left(\int_0^T |e^{-\gamma t}|^2 \|w(t)\|^2_{V(t)} \right)^{\frac{1}{2}} \\
&\leq \|L\dot{w}\|_{L^2_{V^*}} + \mu \|w\|_{L^2_V} \\
&\leq C_{10} \|w\|_{W(V,V^*)} \qquad \text{(by (L1))}
\end{aligned}$$

we end up with

$$\langle L\dot{w}, v_w\rangle_{L^2_{V^*},L^2_V} + \langle Aw, v_w\rangle_{L^2_{V^*},L^2_V} + \langle \Lambda w, v_w\rangle_{L^2_{V^*},L^2_V} \geq C \|w\|_{W(V,V^*)} \|v_w\|_{L^2_V}.$$

$\square$

Lemma 1.5.6. If given arbitrary $v \in L^2_V$, the equality

$$\langle L\dot{w}, v\rangle_{L^2_{V^*},L^2_V} + \langle Aw, v\rangle_{L^2_{V^*},L^2_V} + \langle \Lambda w, v\rangle_{L^2_{V^*},L^2_V} = 0 \tag{1.32}$$

holds for all $w \in W_0(V, V^*)$, then necessarily $v = 0$.

Proof. Define the operator $\tilde{A}(t)\colon V(t) \to V^*(t)$ by

$$\langle \tilde{A}(t)v(t), \eta(t)\rangle_{V^*(t),V(t)} := \langle A(t)\eta(t), v(t)\rangle_{V^*(t),V(t)}$$

and identify $(\tilde{A}v)(t) = \tilde{A}(t)v(t)$. Take $w = \eta \in \mathcal{D}_V$ in (1.32) and rearrange to give

$$\begin{aligned}(L\dot{\eta}, v)_{L^2_H} = (Lv, \dot{\eta})_{L^2_H} &= -\langle \tilde{A}v, \eta\rangle_{L^2_{V^*},L^2_V} - \langle \Lambda v, \eta\rangle_{L^2_{V^*},L^2_V} \\ &= -\langle \tilde{A}v - \Lambda Lv + \Lambda v, \eta\rangle_{L^2_{V^*},L^2_V} - \langle \Lambda Lv, \eta\rangle_{L^2_{V^*},L^2_V}\end{aligned}$$

where we used the symmetric property of $L(t)$. (We could not simply have used A in place of $\tilde{A}$ above because $a(t;\cdot,\cdot)$ may not be symmetric.) This tells us that $\partial^\bullet(Lv) = \tilde{A}v - \Lambda Lv + \Lambda v \in L^2_{V^*}$, and so $Lv \in W(V, V^*)$ (we already have $Lv \in L^2_V$ from (L5)). So

$$\langle \partial^\bullet(Lv), \eta\rangle_{L^2_{V^*},L^2_V} = \langle (\tilde{A} - \Lambda L + \Lambda)v, \eta\rangle_{L^2_{V^*},L^2_V} \qquad \forall \eta \in \mathcal{D}_V.$$

By the density of $\mathcal{D}((0,T); V_0) \subset L^2(0,T; V_0)$, we have the density of $\mathcal{D}_V \subset L^2_V$, which implies

$$\langle \partial^\bullet(Lv), w\rangle_{L^2_{V^*},L^2_V} = \langle (\tilde{A} - \Lambda L + \Lambda)v, w\rangle_{L^2_{V^*},L^2_V} \qquad \forall w \in L^2_V. \tag{1.33}$$

If in particular $w \in W_0(V, V^*)$, then we can use (1.32) on the right hand side of (1.33) to give

$$\langle L\dot{w}, v\rangle_{L^2_{V^*},L^2_V} + \langle \partial^\bullet(Lv), w\rangle_{L^2_{V^*},L^2_V} + \langle \Lambda w, Lv\rangle_{L^2_{V^*},L^2_V} = 0 \quad \forall w \in W_0(V, V^*). \tag{1.34}$$

Using $(L(t)w(t), v(t))_{H(t)} = (L(t)v(t), w(t))_{H(t)}$, we have

$$\begin{aligned}\frac{d}{dt}(L(t)w(t), v(t))_{H(t)} = \langle \partial^\bullet(L(t)v(t)), w(t)\rangle_{V^*(t),V(t)} &+ \langle \dot{w}(t), L(t)v(t)\rangle_{V^*(t),V(t)} \\ &+ \langle \Lambda(t)w(t), L(t)v(t)\rangle_{H^*(t),H(t)}\end{aligned}$$

to which an application of (L2) shows us that (1.34) is exactly

$$\int_0^T \frac{d}{dt}(L(t)w(t), v(t))_{H(t)} = (L(T)w(T), v(T))_{H(T)} = 0$$

for all $w \in W_0(V, V^*)$. Thus we have shown that $v(T) = 0$.

Let $0 > \gamma \in \mathbb{R}$ and set $w(t) = v_\gamma(t) = e^{-\gamma t}v(t)$ in (1.33) to obtain

$$0 = \langle \partial^\bullet(Lv), v_\gamma \rangle_{L^2_{V^*}, L^2_V} - \langle (\tilde{A} - \Lambda L + \Lambda)v, v_\gamma \rangle_{L^2_{V^*}, L^2_V}. \tag{1.35}$$

We showed that $Lv \in W(V, V^*)$ earlier; by (L6), $v \in W(V, V^*)$ too, and so we can apply (L7) to the first term on the right hand side of (1.35):

$$\begin{aligned}
\langle \partial^\bullet(Lv), v_\gamma \rangle_{L^2_{V^*}, L^2_V} &= \langle \dot{L}v, v_\gamma \rangle_{L^2_{V^*}, L^2_V} + \langle L\dot{v}, v_\gamma \rangle_{L^2_{V^*}, L^2_V} \\
&= \langle \dot{L}v, v_\gamma \rangle_{L^2_{V^*}, L^2_V} + \frac{1}{2}\left(\langle L\dot{v}, v_\gamma \rangle_{L^2_{V^*}, L^2_V} + \langle L\dot{v}_\gamma, v \rangle_{L^2_{V^*}, L^2_V} \right) \\
&\quad + \frac{1}{2}\gamma(Lv, v_\gamma)_{L^2_H} \qquad \text{(follows like the equation (1.30))} \\
&\leq \frac{1}{2}\langle \dot{L}v, v_\gamma \rangle_{L^2_{V^*}, L^2_V} - \frac{1}{2}\langle \Lambda v_\gamma, Lv \rangle_{L^2_{V^*}, L^2_V} + \frac{1}{2}\gamma(Lv, v_\gamma)_{L^2_H}. \\
&\qquad \text{(since } v(T) = 0 \text{ and by coercivity of } L(0))
\end{aligned}$$

Note that (L8) together with Young's inequality implies

$$\begin{aligned}
\langle \dot{L}(t)v(t), v(t) \rangle_{V^*(t), V(t)} \leq \|\dot{L}(t)v(t)\|_{V^*(t)} \|v(t)\|_{V(t)} &\leq C_5 \|v(t)\|_{H(t)} \|v(t)\|_{V(t)} \\
&\leq C_\epsilon \|v(t)\|^2_{H(t)} + \epsilon \|v(t)\|^2_{V(t)}.
\end{aligned}$$

Using this and the previous inequality, (1.35) becomes

$$\begin{aligned}
0 &\leq \langle \dot{L}v, v_\gamma \rangle_{L^2_{V^*}, L^2_V} + \langle \Lambda v_\gamma, Lv \rangle_{L^2_{V^*}, L^2_V} + \gamma(Lv, v_\gamma)_{L^2_H} - 2\langle (\tilde{A} + \Lambda)v, v_\gamma \rangle_{L^2_{V^*}, L^2_V} \\
&= \int_0^T e^{-\gamma t} \langle \dot{L}(t)v(t), v(t) \rangle_{V^*(t), V(t)} + \int_0^T e^{-\gamma t} \lambda(t; L(t)v(t), v(t)) \\
&\quad + \int_0^T \gamma e^{-\gamma t} (L(t)v(t), v(t))_{H(t)} - 2\int_0^T e^{-\gamma t} \langle (\tilde{A}(t) + \Lambda(t))v(t), v(t) \rangle_{V^*(t), V(t)} \\
&\leq (C_1 + \gamma C_2) \int_0^T e^{-\gamma t} \|v(t)\|^2_{H(t)} - 2C_a \int_0^T e^{-\gamma t} \|v(t)\|^2_{V(t)}
\end{aligned}$$

using the bound on $\lambda(t; \cdot, \cdot)$ and the assumptions (L3), (L4) and (A1) (coercivity). If we pick $\gamma = -\frac{C_1}{C_2}$, it follows that $v = 0$ in L^2_V. $\square$

Proof of Theorem 1.4.1. The inf-sup condition (which is an easy consequence of Lemma 1.5.5) in combination with Lemma 1.5.6 furnishes the requirements of the

Banach–Nečas–Babuška theorem (Theorem 1.5.1) thus yielding the existence and uniqueness of a solution $w \in W_0(V, V^*)$ to

$$\begin{aligned} L\dot{w} + Aw + \Lambda w &= \tilde{f} \\ w(0) &= 0 \end{aligned}$$

where $\tilde{f} \in L^2_{V^*}$ is arbitrary. Hence, we have well-posedness of $(\mathbf{P_0})$ with the estimate

$$\|w\|_{W(V,V^*)} \leq C\|\tilde{f}\|_{L^2_{V^*}}.$$

From this well-posedness result, we also obtain unique solvability of $(\mathbf{P})$ by setting $u = w + \tilde{y}$ (note that w depends on $\tilde{y}$), with the solution $u \in W(V, V^*)$ satisfying

$$\|u\|_{W(V,V^*)} \leq C\left(\|f\|_{L^2_{V^*}} + \|u_0\|_{H_0}\right).$$

□

1.6 Galerkin approximation

In this section we abstract the pushed-forward Galerkin method used in [48] for the advection-diffusion equation on an evolving hypersurface.

1.6.1 Finite-dimensional spaces

Let $\{\chi_j^0\}_{j\in\mathbb{N}}$ be the basis of V_0 described in Assumption 1.4.2 and recall that $\chi_j^t := \phi_t(\chi_j^0)$. The next lemma follows easily.

Lemma 1.6.1. The set $\{\chi_j^t\}_{j\in\mathbb{N}}$ is a countable basis of $V(t)$.

The next result is an extremely useful property of the basis functions that follows from Remark 1.2.32 (see [48] for the finite element analogue).

Lemma 1.6.2 (Transport property of basis functions)**.** The basis $\{\chi_j^t\}_{j\in\mathbb{N}}$ satisfies the transport property

$$\dot{\chi}_j^t = 0.$$

We now construct the approximation spaces in which the discrete solutions lie.

Definition 1.6.3 (Approximation spaces)**.** For each $N \in \mathbb{N}$ and each $t \in [0, T]$, define

$$V_N(t) = \operatorname{span}\{\chi_1^t, ..., \chi_N^t\} \subset V(t).$$

Clearly $V_N(t) \subset V_{N+1}(t)$ and $\bigcup_{j\in\mathbb{N}} V_j(t)$ is dense in $V(t)$. Define

$$L^2_{V_N} = \{u \in L^2_V \mid u(t) = \sum_{j=1}^{N} \alpha_j(t)\chi_j^t \text{ where } \alpha_j \colon [0,T] \to \mathbb{R}\}.$$

Similarly, $L^2_{V_N} \subset L^2_{V_{N+1}}$, and we shall state a density result below which follows from the density of the embedding $\bigcup_{j\in\mathbb{N}} L^2(0,T;V_j(0)) \subset L^2(0,T;V_0)$ and from the fact that $L^2(0,T;V_j(0)) \subset L^2(0,T;V_{j+1}(0))$.

Lemma 1.6.4. The space $\bigcup_{j\in\mathbb{N}} L^2_{V_j}$ is dense in L^2_V.

Remark 1.6.5. If $u \in L^2_{V_N}$ and $u(t) = \sum_{j=1}^N \alpha_j(t)\chi_j^t$ with $\alpha_j \in C^1([0,T])$, then $u \in C^1_V$ with strong material derivative $\dot{u}(t) = \sum_{j=1}^N \alpha_j'(t)\chi_j^t$, and $\dot{u} \in L^2_{V_N}$. Our Galerkin ansatz (see below) has coefficients in a slightly less convenient space.

Galerkin ansatz. Later on, we construct finite-dimensional solutions which have the form

$$u_N(t) = \sum_{j=1}^{N} u_j^N(t)\chi_j^t \in V_N(t)$$

where the $u_j^N \colon [0,T] \to \mathbb{R}$ turn out to be absolutely continuous coefficient functions with $\dot{u}_j^N \in L^2(0,T)$, i.e., $u_N \in \tilde{C}^1_V$. It holds that $u_N \in L^2_V$ and by definition, $u_N \in L^2_{V_N}$. By Remark 1.4.5, the material derivative of u_N is $\dot{u}_N \in L^2_{V_N}$ with $\dot{u}_N(t) = \sum_{j=1}^N \dot{u}_j^N(t)\chi_j^t$.

Definition 1.6.6 (Projection operators)**.** For each $t \in [0,T]$, define a projection operator $P_N^t \colon H(t) \to V_N(t)$ by the formula

$$(P_N^t u - u, v_N)_{H(t)} = 0 \quad \text{for all } v_N \in V_N(t).$$

It follows that $(P_N^t)^2 = P_N^t$,

$$\left\|P_N^t u\right\|_{H(t)} \le \|u\|_{H(t)}$$

and

$$P_N^t u \to u \quad \text{in } H(t) \text{ for all } u \in H(t). \tag{1.36}$$

It is worth emphasising that our operators P_N^t are defined as Hilbert space projections using the inner product without reference to an orthonormal eigenbasis. That is, they are not defined by truncating an infinite eigenbasis expansion (except when $t = 0$). This is because the pushforward basis elements $\{\chi_j^t\}_{j\in\mathbb{N}}$ are not necessarily orthonormal in $H(t)$ for $t > 0$ (this would have been a convenient property to have but we persist with the χ_j^t because they have the transport property $\dot{\chi}_j^t = 0$).

Remark 1.6.7. We could have relaxed the definition of the spaces $V_N(t)$ and instead have asked for a family of finite-dimensional spaces $\{V_N(0)\}_{N\in\mathbb{N}}$ such that for all $N \in \mathbb{N}$,

(i) $V_N(0) \subset V_0$

(ii) $\dim(V_N) = N$

(iii) $\bigcup_{i\in\mathbb{N}} V_i(0)$ is dense in V_0

(iv) For every $v \in V_0$, there exists a sequence $\{v_N\}_{N\in\mathbb{N}}$ with $v_N \in V_N(0)$ such that $\|v_N - v\|_{V_0} \to 0$.

Furthermore, we can define the spaces $V_N(t) := \phi_t(V_N(0))$. The continuity of the map ϕ_t implies that these spaces share the same properties (with respect to $V(t)$) as the $V_N(0)$ given above; in particular the density result

$$\bigcup_{N\in\mathbb{N}} V_N(t) \quad \text{is dense in } V(t)$$

is true. Note that the basis of $V_N(t)$ does not necessarily have to be a subset of the basis of $V_{N+1}(t)$; this is the situation in finite element analysis, for example, so this relaxation can be useful for the purposes of numerical analysis. See [48], [49].

1.6.2 Galerkin approximation of (P)

We now proceed with the regularity result. With $f \in L^2_H$ and $u_0 \in V_0$, the finite-dimensional approximation is to find a unique $u_N \in L^2_{V_N}$ with $\dot{u}_N \in L^2_{V_N}$ satisfying

$$\begin{aligned} l(t; \dot{u}_N(t), \chi_j^t) + a(t; u_N(t), \chi_j^t) + \lambda(t; u_N(t), \chi_j^t) &= (f(t), \chi_j^t)_{H(t)} \\ u_N(0) &= u_{0N} \end{aligned} \tag{1.37}$$

for all $j \in \{1, ..., N\}$ and for almost every $t \in [0, T]$ (cf. the equation (**P'**)). Here, u_{0N} is as in Assumption 1.4.2.

Theorem 1.6.8 (Well-posedness of solutions to the finite-dimensional problem)**.** Under the hypotheses of Theorem 1.4.8, there exists a unique $u_N \in L^2_{V_N}$ with $\dot{u}_N \in$

$L^2_{V_N}$ satisfying the finite-dimensional problem (1.37). With $u_N(t) = \sum_{i=1}^N u_i^N(t)\chi_i^t$, the coefficient functions satisfy

$$u_i^N \in AC([0,T])$$
$$\dot{u}_i^N \in L^2(0,T).$$

for all $i \in \{1, ..., N\}$.

Proof. Substitute $u_N(t) = \sum_{i=1}^N u_i^N(t)\chi_i^t$ into (1.37) to yield

$$\sum_{i=1}^N \dot{u}_i^N(t)l_{ij}(t) + u_i^N(t)(a_{ij}(t) + c_{ij}(t)) = f_j(t) \tag{1.38}$$

with $l_{ij}(t) = l(t;\chi_i^t,\chi_j^t)$, $a_{ij}(t) = a(t;\chi_i^t,\chi_j^t)$, $\lambda_{ij}(t) = \lambda(t;\chi_i^t,\chi_j^t)$ and $f_j(t) = (f(t),\chi_j^t)_{H(t)}$. Defining the vectors $(\mathbf{u^N(t)})_i = u_i^N(t)$ and $(\mathbf{F(t)})_i = f_i(t)$, and matrices $(\mathbf{L(t)})_{ij} = l_{ji}(t)$, $(\mathbf{A(t)})_{ij} = a_{ji}(t)$, and $(\mathbf{\Lambda(t)})_{ij} = \lambda_{ji}(t)$, we can write (1.38) in matrix-vector form as

$$\mathbf{L(t)\dot{u}^N(t) + (A(t) + \Lambda(t))u^N(t) = F(t)}.$$

Elementary considerations show that $\mathbf{L(t)}^{-1}$ exists with $\mathbf{L}(\cdot)^{-1} \in L^\infty(0,T;\mathbb{R}^{N\times N})$, so we can rearrange the system to

$$\mathbf{\dot{u}^N(t) + L(t)^{-1}(A(t) + \Lambda(t))u^N(t) = L(t)^{-1}F(t)}. \tag{1.39}$$

Note that $\mathbf{F}(\cdot) \in L^2(0,T;\mathbb{R}^N)$ and $\mathbf{A}(\cdot) + \mathbf{\Lambda}(\cdot) \in L^\infty(0,T;\mathbb{R}^{N\times N})$. So the coefficients of (1.39) are all measurable in time, and we can apply standard theory that guarantees the existence and uniqueness of $u_j^N \in AC([0,T])$ with $\dot{u}_j^N \in L^2(0,T)$, and thus the existence and uniqueness of u_N. The function $u_N \in \tilde{C}^1_V$ is a solution in the sense that the derivative $\dot{u}_N$ exists almost everywhere and the ODE is satisfied almost everywhere. $\square$

The Galerkin approximation is equivalent to the discrete equation

$$l(t;\dot{u}_N(t),v_N(t)) + a(t;u_N(t),v_N(t)) + \lambda(t;u_N(t),v_N(t)) = (f(t),v_N(t))_{H(t)} \tag{$\mathbf{P'_d}$}$$

for all $v_N \in L^2_{V_N}$. We look for *a priori* estimates on u_N and $\dot{u}_N$ in appropriate norms.

Lemma 1.6.9 (A priori estimate on u_N)**.** Under the hypotheses of Theorem 1.4.8, the following estimate holds:

$$\|u_N\|_{L^2_V} \leq C \left(\|u_0\|_{H_0} + \|f\|_{L^2_{V^*}} \right).$$

Remark 1.6.10. This *a priori* estimate is still valid under the hypotheses of Theorem 1.4.1 if we pick $u_N(0)$ differently. See §1.6.4 for more.

Proof of Lemma 1.6.9. Picking $v_N = u_N$ in ($\mathbf{P'_d}$) gives

$$l(t; \dot{u}_N, u_N) + a(t; u_N, u_N) + \lambda(t; u_N, u_N) = (f, u_N)_{H(t)},$$

which we integrate in time and apply the transport identity (1.26) to yield

$$\begin{aligned}\int_0^T \frac{1}{2}\frac{d}{dt} l(t; u_N, u_N) + a(t; u_N, u_N) + \lambda(t; u_N, u_N) - \frac{1}{2} m(t; u_N, u_N)&\\ = \int_0^T (f, u_N)_{H(t)}.&\end{aligned}$$

Using the boundedness (L3) and coercivity (L4) of $l(t; \cdot, \cdot)$ leads to

$$\begin{aligned}\frac{C_c}{2}\|u_N(T)\|^2_{H(T)} + \int_0^T a(t; u_N, u_N) + \int_0^T \lambda(t; u_N, u_N) - \frac{1}{2}\int_0^T m(t; u_N, u_N)&\\ \leq \int_0^T \langle f, u_N \rangle_{V^*(t), V(t)} + \frac{C_b}{2}\|u_N(0)\|^2_{H_0},&\end{aligned}$$

to which we use (A1) (the coercivity of $a(t; \cdot, \cdot)$), the boundedness of $\lambda(t; \cdot, \cdot)$ and $m(t; \cdot, \cdot)$, and Young's inequality with $\epsilon > 0$:

$$\begin{aligned}\frac{C_c}{2}\|u_N(T)\|^2_{H(T)} + \frac{C_1}{2}\|u_N\|^2_{L^2_V} \leq \frac{C_2}{2}\|u_N\|^2_{L^2_H} + \frac{1}{2\epsilon}\|f\|^2_{L^2_{V^*}} + \frac{\epsilon}{2}\|u_N\|^2_{L^2_V}&\\ + \frac{C_b}{2}\|u_N(0)\|^2_{H_0}.&\end{aligned}$$

That is,

$$C_c\|u_N(T)\|^2_{H(T)} + (C_1 - \epsilon)\|u_N\|^2_{L^2_V} \leq \frac{1}{\epsilon}\|f\|^2_{L^2_{V^*}} + C_2\|u_N\|^2_{L^2_H} + C_b\|u_N(0)\|^2_{H_0}, \quad (1.40)$$

and if ϵ is picked small enough, we can discard the second term on the left hand side and then an application of Gronwall's inequality yields

$$\|u_N(t)\|^2_{H(t)} \leq C_4 \left(\|f\|^2_{L^2_{V^*}} + \|u_N(0)\|^2_{H_0} \right).$$

Using this on (1.40) and utilising (B2) produces the desired estimate. □

Lemma 1.6.11 (A priori estimate on $\dot{u}_N$)**.** Under the hypotheses of Theorem 1.4.8, the following estimate holds:

$$\|\dot{u}_N\|_{L^2_H} \leq C\left(\|u_0\|_{V_0} + \|f\|_{L^2_H}\right).$$

Proof. In ($\mathbf{P'_d}$), pick $v_N = \dot{u}_N$ and use (L4) to get

$$C_1 \|\dot{u}_N\|^2_{H(t)} + a_s(t; u_N, \dot{u}_N) + a_n(t; u_N, \dot{u}_N) + \lambda(t; u_N, \dot{u}_N) \leq (f, \dot{u}_N)_{H(t)}. \quad (1.41)$$

Then using assumption (A7), (1.41) is

$$\begin{aligned} C_1 \|\dot{u}_N\|^2_{H(t)} + \frac{1}{2}\frac{d}{dt} a_s(t; u_N, u_N) \leq (f, \dot{u}_N)_{H(t)} + \frac{1}{2} r(t; u_N) - a_n(t; u_N, \dot{u}_N) \\ - \lambda(t; u_N, \dot{u}_N). \end{aligned}$$

Integrating this yields

$$\begin{aligned} &C_1 \int_0^T \|\dot{u}_N\|^2_{H(t)} + \frac{1}{2} a_s(T; u_N(T), u_N(T)) \\ &\leq \int_0^T (f, \dot{u}_N)_{H(t)} + \frac{1}{2}\int_0^T r(t; u_N) - \int_0^T a_n(t; u_N, \dot{u}_N) - \int_0^T \lambda(t; u_N, \dot{u}_N) \\ &\quad + \frac{1}{2} a_s(0; u_N(0), u_N(0)). \end{aligned}$$

where we used (A3). With (A6) (positivity of $a_s(t; \cdot, \cdot)$), the bound (A5) on $a_s(0; \cdot, \cdot)$, the bound (A8) on $r(t; \cdot)$, the bound (A4) on $a_n(t; \cdot, \cdot)$, the bound on $\lambda(t; \cdot, \cdot)$ and Young's inequality with $\epsilon > 0$ and $\delta > 0$, we get

$$\begin{aligned} C_1 \|\dot{u}_N\|^2_{L^2_H} &\leq \frac{1}{2\delta}\|f\|^2_{L^2_H} + \left(C_2 + \frac{C_3}{2\epsilon}\right)\|u_N\|^2_{L^2_V} + \frac{(\delta + C_3\epsilon)}{2}\|\dot{u}_N\|^2_{L^2_H} \\ &\quad + C_4 \|u_N(0)\|^2_{V_0} \\ &\leq \frac{1}{2\delta}\|f\|^2_{L^2_H} + C_5\left(C_2 + \frac{C_3}{2\epsilon}\right)(\|u_N(0)\|^2_{H_0} + \|f\|^2_{L^2_H}) \\ &\quad + \frac{(\delta + C_3\epsilon)}{2}\|\dot{u}_N\|^2_{L^2_H} + C_4\|u_N(0)\|^2_{V_0} \quad \text{(by the first } \textit{a priori} \text{ bound)} \\ &= \left(\frac{1}{2\delta} + C_5\left(C_2 + \frac{C_3}{2\epsilon}\right)\right)\|f\|^2_{L^2_H} + C_5\left(C_2 + \frac{C_3}{2\epsilon}\right)\|u_N(0)\|^2_{H_0} \\ &\quad + \frac{(\delta + C_3\epsilon)}{2}\|\dot{u}_N\|^2_{L^2_H} + C_4\|u_N(0)\|^2_{V_0}. \end{aligned}$$

If ϵ and δ are small, we can obtain the estimate by using the assumption (B3). $\square$

1.6.3 Proof of regularity

By the estimates above, we obtain the convergence

$$\begin{aligned} u_N &\rightharpoonup u \quad \text{in } L^2_V \\ \dot{u}_N &\rightharpoonup w \quad \text{in } L^2_H \end{aligned} \tag{1.42}$$

for some $u \in L^2_V$ and $w \in L^2_H$ and for a subsequence which we have relabelled. Now we show that in fact, $w = \dot{u}$.

Lemma 1.6.12. In the context of the above convergence results, $w = \dot{u}$.

Proof. By definition

$$\int_0^T \langle \dot{u}_N(t), \eta(t)\rangle_{V^*(t),V(t)} = -\int_0^T (u_N(t), \dot{\eta}(t))_{H(t)} - \int_0^T \lambda(t; u_N(t), \eta(t)) \tag{1.43}$$

holds for all $\eta \in \mathcal{D}_V(0,T)$. Since $\langle \cdot, \eta\rangle_{L^2_{V^*}, L^2_V}$, $(\cdot, \dot{\eta})_{L^2_H}$, and $\langle \Lambda(\cdot), \eta\rangle_{L^2_{V^*}, L^2_V}$ are all elements of $L^2_{V^*}$, using (1.42), we can pass to the limit in (1.43) to obtain

$$\int_0^T \langle w(t), \eta(t)\rangle_{V^*(t),V(t)} = -\int_0^T (u(t), \dot{\eta}(t))_{H(t)} - \int_0^T \lambda(t; u(t), \eta(t)),$$

i.e., $w = \dot{u}$. □

Proof of Theorem 1.4.8. Given $v \in L^2_V$, by density, there is a sequence $\{v_M\}$ with $v_M \in L^2_{V_M}$ for each M such that

$$v_M(t) = \sum_{j=1}^M \alpha_j^M(t)\chi_j^t \qquad \text{and} \qquad \|v_M - v\|_{L^2_V} \to 0.$$

For $j = 1, ..., N$, consider the equation (1.37):

$$l(t; \dot{u}_N(t), \chi_j^t) + a(t; u_N(t), \chi_j^t) + \lambda(t; u_N(t), \chi_j^t) = (f(t), \chi_j^t)_{H(t)}.$$

If $M \le N$, then $v_M \in L^2_{V_N}$ and we multiply the above by $\alpha_j^M(t)$ and sum up to get

$$l(t; \dot{u}_N(t), v_M(t)) + a(t; u_N(t), v_M(t)) + \lambda(t; u_N(t), v_M(t)) = (f(t), v_M(t))_{H(t)}.$$

By the bounds on the respective bilinear forms, we see that $\langle L(\cdot), v_M\rangle$, $\langle A(\cdot), v_M\rangle$, and $\langle \Lambda(\cdot), v_M\rangle$ are elements of $L^2_{V^*}$, so we obtain after integrating the above equation and taking the limit as $N \to \infty$ the equation

$$\int_0^T l(t;\dot{u}(t), v_M(t)) + a(t; u(t), v_M(t)) + \lambda(t; u(t), v_M(t)) = \int_0^T (f(t), v_M(t))_{H(t)}.$$

Now note that as a function of v_M, each term in the above equation is an element of $L^2_{V^*}$ again because of the bounds on $l(t;\cdot,\cdot)$, $a(t;\cdot,\cdot)$ and $\lambda(t;\cdot,\cdot)$. So we send $M \to \infty$, bearing in mind that v_M strongly converges to v in L^2_V:

$$\int_0^T l(t;\dot{u}(t), v(t)) + a(t; u(t), v(t)) + \lambda(t; u(t), v(t)) = \int_0^T (f(t), v(t))_{H(t)}.$$

Hence $u \in W(V, H)$ is a solution. Let us now check the initial condition. Let $w \in V_0$, take $\zeta \in C^1[0, T]$ with $\zeta(T) = 0$, and set $v(t) = \zeta(t)\phi_t w$; we see that $v \in L^2_V$. Since $w \in V_0$, there exist coefficients α_j with $w = \sum_{j=1}^\infty \alpha_j \chi_j^0$, so

$$v(t) = \zeta(t) \sum_{j=1}^\infty \alpha_j \chi_j^t. \tag{1.44}$$

The sequence $\{v_N\}_{N\in\mathbb{N}}$ defined by

$$v_N(t) = \zeta(t) \sum_{j=1}^N \alpha_j \chi_j^t \tag{1.45}$$

is such that $v_N \in L^2_{V_N}$ and satisfies $\|v_N - v\|_{L^2_V} \to 0$ by definition of w as an infinite sum. Similarly, we can show that $\dot{v}_N \to \dot{v}$ in L^2_V. Using the identity (1.26) with v chosen as in (1.44), we see that

$$\begin{aligned} -l(0; u(0), v(0)) + &\int_0^T a(t; u(t), v(t)) + \lambda(t; u(t), v(t)) \\ &= \int_0^T (f(t), v(t))_{H(t)} + l(t; u(t), \dot{v}(t)) + m(t; u(t), v(t)). \end{aligned} \tag{1.46}$$

Similarly, with v_N chosen as in (1.45) in the Galerkin equation ($\mathbf{P'_d}$), to which we again apply (1.26) and integrate to obtain

$$\begin{aligned} -l(0; u_N(0), v_N(0)) + &\int_0^T a(t; u_N(t), v_N(t)) + \lambda(t; u_N(t), v_N(t)) \\ &= \int_0^T (f(t), v_N(t))_{H(t)} + l(t; u_N(t), \dot{v}_N(t)) + m(t; u_N(t), v_N(t)). \end{aligned}$$

Using $u_N \rightharpoonup u$, $v_N \to v$, $\dot{v}_N \to \dot{v}$, and (B1), we may pass to the limit in this equation and a comparison of the result to (1.46) will tell us that

$$l(0; u_0 - u(0), \zeta(0)w) = 0.$$

The arbitrariness of $w \in V_0$ and the density of V_0 in H_0 yield the result.

The stability estimate follows directly from the estimates in Lemmas 1.6.9 and 1.6.11. That the solution is unique follows by a straightforward adaptation of the standard technique.

□

1.6.4 Second sketch proof of existence

Sketch proof of Theorem 1.4.1. We can take the Galerkin approximation of (1.28) and instead of picking the initial data of u_N to be u_{0N} we pick $u_N(0) = P_N^0(u_0)$, where P_N^0 is the projection operator in Definition 1.6.6. We still obtain the uniform bound of Lemma 1.6.9, which implies that

$$u_N \rightharpoonup u \quad \text{in } L_V^2 \tag{1.47}$$

for some $u \in L_V^2$. An equation similar to $(\mathbf{P'_d})$ will hold, in which we pick $v_N(t) = \chi_j^t$, where $j \in \{0, ..., N\}$, and multiplying by $\zeta \in C^1[0, T]$ with $\zeta(T) = 0$, we get

$$l(t; \dot{u}_N, \zeta\chi_j) + a(t; u_N, \zeta\chi_j) + \lambda(t; u_N, \zeta\chi_j) = \langle f, \zeta\chi_j \rangle_{V^*(t), V(t)},$$

and then integrating, using the transport formula (1.26), and passing to the limit with the help of (1.47) and (1.36):

$$\begin{aligned} -\int_0^T l(t; u(t), \zeta'(t)\chi_j^t) + a(t; u(t), \zeta(t)\chi_j^t) + \lambda(t; u(t), \zeta(t)\chi_j^t) - m(t; u(t), \zeta(t)\chi_j^t) \\ = \int_0^T \langle f(t), \zeta(t)\chi_j^t \rangle_{V^*(t), V(t)} + l(0; u_0, \zeta(0)\chi_j^0). \end{aligned} \tag{1.48}$$

Now, we can write an arbitrary element of V_0 as $v = \sum_{i=1}^{\infty} \alpha_j \chi_j^0$. By definition, the sequence $v_n = \sum_{i=1}^{n} \alpha_j \chi_j^0$ converges to v in V_0. It follows that $\phi_t v_n \to \phi_t v$ in $V(t)$. Letting $\zeta(0) = 0$, multiplying (1.48) by α_j and summing over j gives us

$$\begin{aligned} &\int_0^T \zeta'(t) l(t; u(t), \phi_t v_n) \\ &\quad = -\int_0^T \zeta(t) \langle f(t) - A(t)u(t) - \Lambda(t)u(t) + M(t)u(t), \phi_t v_n \rangle_{V^*(t), V(t)}. \end{aligned} \tag{1.49}$$

It is not difficult to see that the dominated convergence theorem applies and we can pass to the limit in (1.49) to obtain

$$\int_0^T \zeta'(t) l(t; u(t), \phi_t v)$$
$$= -\int_0^T \zeta(t) \langle f(t) - A(t)u(t) - \Lambda(t)u(t) + M(t)u(t), \phi_t v \rangle_{V^*(t), V(t)}.$$

If we further let $\zeta \in \mathcal{D}(0,T)$, this is precisely the statement

$$\frac{d}{dt} l(t; u(t), \phi_t v) = \langle f(t) - A(t)u(t) - \Lambda(t)u(t) + M(t)u(t), \phi_t v \rangle_{V^*(t), V(t)}$$

in the weak sense. This is true for every $v \in V_0$, and because $f - Au - \Lambda u \in L^2_{V^*}$, by Lemma 1.3.5, $L\dot{u} + A + \Lambda u = f$ holds as an equality in $L^2_{V^*}$ with $u \in W(V, V^*)$. $\square$

Chapter 2

Applications of the abstract framework to evolving surfaces and domains

2.1 Introduction

The purpose of this chapter is twofold: first, to give an account of how the abstract framework that we developed above to handle linear parabolic equations on *abstract* evolving Hilbert spaces can be applied to the case of Lebesgue–Sobolev–Bochner spaces on moving hypersurfaces (and domains), and second, to use the power of this framework to study four different parabolic equations posed on moving hypersurfaces. We begin by starting with a surface heat equation on an evolving compact hypersurface without boundary, and the following on an evolving domain: a bulk equation, a coupled bulk-surface system and a problem with a dynamic boundary condition. The first three problems are relevant to physical applications and the last problem is more of a toy model which nonetheless is extremely useful for a later application in Chapter 4; we will motivate these problems in greater detail later.

To formulate these problems, we obviously first need to discuss hypersurfaces and Sobolev spaces defined on hypersurfaces. For reasons of space we shall only briefly touch upon the theory here in §2.2 and refer the reader to [50, 42, 127, 70, 109] for more details on analysis on surfaces; we emphasise the text [109] which contains a detailed overview of the essential facts.

We shall give the equations we wish to study in §2.3. In §2.4, we discuss in detail realisations of the abstraction to the concrete case of moving domains (which are a special case of evolving flat hypersurfaces) and evolving curved hypersurfaces, i.e., we show that the abstract framework is applicable for moving hypersurfaces. Then, we finish in §2.5 by proving the well-posedness of the four problems introduced in §2.3.

2.2 Evolving hypersurfaces and Sobolev spaces

Hypersurfaces Recall that Γ is an n-dimensional C^k *hypersurface* in $\mathbb{R}^{n+1}$ if for each $x \in \Gamma$, there is an open set $U \subset \mathbb{R}^{n+1}$ with $x \in U$ and a function $\Psi \in C^k(U)$ with $\nabla\Psi \neq 0$ on $\Gamma \cap U$ and

$$\Gamma \cap U = \{x \in U \mid \Psi(x) = 0\}.$$

A *parametrised* C^k *hypersurface* in $\mathbb{R}^{n+1}$ is a map $\psi \in C^k(Y; \mathbb{R}^{n+1})$ where $Y \subset \mathbb{R}^n$ is a connected open set with $\text{rank}(D\psi(y)) = n$ for all $y \in Y$. Locally, parametrised hypersurfaces and hypersurfaces are the same [119, Chapter 15]. We call Γ a C^k *hypersurface with boundary* $\partial\Gamma$ if $\Gamma\backslash\partial\Gamma$ is a C^k hypersurface and if for every $x \in \partial\Gamma$, there exists an open set $U \subset \mathbb{R}^{n+1}$ with $x \in U$ and a homeomorphism $\psi\colon H \to \Gamma\cap U$, where $H := B_1(0) \cap \{y = (y_1, ..., y_n) \in \mathbb{R}^n \mid y_n \leq 0\}$, with $\psi(0) = x$ and

1. $\text{rank}(D\psi(y)) = n$ for all $y \in H$
2. $\psi(B_1(0) \cap \{y = (y_1, ..., y_n) \in \mathbb{R}^n \mid y_n < 0\}) \subset \Gamma\backslash\partial\Gamma$
3. $\psi(B_1(0) \cap \{y = (y_1, ..., y_n) \in \mathbb{R}^n \mid y_n = 0\}) \subset \partial\Gamma$.

See [119, Chapter 20]. A *compact hypersurface* has no boundary. We say Γ is a *compact hypersurface with boundary* $\partial\Gamma$ if Γ is a hypersurface with boundary $\partial\Gamma$ and $\Gamma \cup \partial\Gamma$ is compact. Throughout this work we assume that Γ is orientable with unit normal ν. We say Γ is *flat* if the normal ν is same everywhere on Γ.

Sobolev spaces Suppose that Γ is an n-dimensional compact C^k hypersurface in $\mathbb{R}^{n+1}$ with $k \geq 2$ and smooth boundary $\partial\Gamma$. We can define $L^2(\Gamma)$ in the natural way: it consists of the set of measurable functions $f\colon \Gamma \to \mathbb{R}$ such that

$$\|f\|_{L^2(\Gamma)} := \left(\int_\Gamma |f(x)|^2 \,\mathrm{d}\sigma(x)\right)^{\frac{1}{2}} < \infty,$$

where $\mathrm{d}\sigma$ is the surface measure on Γ (which we often omit). We will use the notation $\nabla_\Gamma = (\underline{D}_1, ..., \underline{D}_{n+1})$ to stand for the surface gradient on a hypersurface Γ, and $\Delta_\Gamma := \nabla_\Gamma \cdot \nabla_\Gamma$ will denote the Laplace–Beltrami operator. The integration by parts formula for functions $f \in C^1(\overline{\Gamma}; \mathbb{R}^{n+1})$ is

$$\int_\Gamma \nabla_\Gamma \cdot f = \int_\Gamma f \cdot H\nu + \int_{\partial\Gamma} f \cdot \mu$$

where H is the mean curvature and μ is the unit conormal vector which is normal to $\partial\Gamma$ and tangential to Γ. Now if $\psi \in C_c^1(\Gamma)$, then this formula implies

$$\int_\Gamma f \underline{D}_i \psi = -\int_\Gamma \psi \underline{D}_i f + \int_\Gamma f\psi H\nu_i \qquad \text{for } i = 1, ..., n+1,$$

with the boundary term disappearing due to the compact support. This relation is the basis for defining weak derivatives. We say $f \in L^2(\Gamma)$ has weak derivative $g_i =: \underline{D}_i f \in L^2(\Gamma)$ if for every $\psi \in C_c^1(\Gamma)$,

$$\int_\Gamma f \underline{D}_i \psi = -\int_\Gamma \psi g_i + \int_\Gamma f\psi H\nu_i$$

holds. Then we can define the Sobolev space

$$H^1(\Gamma) = \{f \in L^2(\Gamma) \mid \underline{D}_i f \in L^2(\Gamma), i = 1, ..., n+1\}$$

with $\|f\|^2_{H^1(\Gamma)} := \|f\|^2_{L^2(\Gamma)} + \|\nabla_\Gamma f\|^2_{L^2(\Gamma)}$. The above applies to compact hypersurfaces too; in this case the boundary terms in the integration by parts are simply not there. We write $H^{-1}(\Gamma)$ for the dual space of $H^1(\Gamma)$ when Γ is a compact hypersurface.

We shall also need a fractional-order Sobolev space. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with boundary $\partial\Omega$. Define the space

$$H^{\frac{1}{2}}(\partial\Omega) = \{u \in L^2(\partial\Omega) \mid \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^n} \,\mathrm{d}\sigma(x)\mathrm{d}\sigma(y) < \infty\}.$$

This is a Hilbert space with the inner product

$$\begin{aligned}(u, v)_{H^{\frac{1}{2}}(\partial\Omega)} = &\int_{\partial\Omega} u(x)v(x) \,\mathrm{d}\sigma(x) \\ &+ \int_{\partial\Omega}\int_{\partial\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^n} \,\mathrm{d}\sigma(x)\mathrm{d}\sigma(y).\end{aligned}$$

See [109, §2.4] and [43, §3.2] for details. The notation

$$|u|_{H^{\frac{1}{2}}(\partial\Omega)} = \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^n} \,\mathrm{d}\sigma(x)\mathrm{d}\sigma(y)\right)^{\frac{1}{2}}$$

for the seminorm is convenient. Now, recall the standard Green's formula:

$$\int_{\partial\Omega} \frac{\partial v}{\partial \nu} w = \int_\Omega \nabla v \nabla w + \int_\Omega w \Delta v \qquad \forall v \in H^2(\Omega),\ \forall w \in H^1(\Omega).$$

When Ω is of class C^1, this formula leads us to define a (weak) normal derivative for functions $v \in H^1(\Omega)$ with $\Delta v \in L^2(\Omega)$ as the element $\partial v/\partial \nu \in H^{-\frac{1}{2}}(\partial\Omega) := (H^{\frac{1}{2}}(\partial\Omega))^*$ determined by

$$\left\langle \frac{\partial v}{\partial \nu}, w \right\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} := \int_\Omega \nabla v \nabla \mathbb{E}(w) + \int_\Omega \mathbb{E}(w) \Delta v \quad \forall w \in H^{\frac{1}{2}}(\partial\Omega), \tag{2.1}$$

where $\mathbb{E}(w) \in H^1(\Omega)$ is an extension of $w \in H^{\frac{1}{2}}(\partial\Omega)$; the functional $\partial v/\partial\nu$ is independent of the extension used for w. See [43, §5.5.1] for more details on this.

Evolving hypersurfaces We say that $\{\Gamma(t)\}_{t\in[0,T]}$ is an *evolving hypersurface* if for every $t_0 \in [0, T]$, there exist open sets $I = (t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$ and $U \subset \mathbb{R}^{n+1}$ and a map $\Psi\colon I \times U \to \mathbb{R}$ such that $\nabla\Psi(t, x) \neq 0$ for $x \in \Gamma(t)$ and $t \in I$, and

$$\Gamma(t) \cap U = \{x \in U \mid \Psi(t, x) = 0\} \quad \text{for } t \in I.$$

The *normal velocity* of a hypersurface $\Gamma(t) := \{x \in \mathbb{R}^{n+1} \mid \Psi(x, t) = 0\}$ defined by a (global) level set function is given by

$$\mathbf{w}_\nu = -\frac{\Psi_t}{|\nabla\Psi|}\frac{\nabla\Psi}{|\nabla\Psi|}.$$

Remark 2.2.1. It is important to note that *the normal velocity is sufficient to define the evolution of a compact hypersurface.* However, a parametrised hypersurface would require the prescription of the full velocity of the parametrisation.

Remark 2.2.2. Consider an evolving hypersurface with boundary. In this case, we need the normal velocity of the surface and the conormal velocity of the boundary in order to describe the evolution. The normal velocity of the surface must agree with the normal velocity of the boundary.

Remark 2.2.3. An evolving bounded domain $\{\Omega(t)\}$ in $\mathbb{R}^n$ can be viewed as an evolving flat hypersurface with boundary $\{\hat{\Omega}(t)\}$ in $\mathbb{R}^{n+1}$ (though we choose not to use this viewpoint in this thesis). If we embed each $\Omega(t)$ into the same hyperplane of $\mathbb{R}^{n+1}$ (for example, $\hat{\Omega}(t) = \{(x_1, ..., x_n, 0) \mid (x_1, ..., x_n) \in \Omega(t)\}$), then the normal velocity $\mathbf{w}_\nu$ of $\hat{\Omega}(t)$ is zero.

In order to describe the evolution of a hypersurface, it is also useful to assume that there exists a map $F(\cdot,t)\colon \Gamma(0) \to \Gamma(t)$ which is a diffeomorphism for each $t \in [0,T]$ satisfying $F(\cdot,0) \equiv \mathrm{Id}$ and $\frac{d}{dt}F(\cdot,t) = \mathbf{w}(F(\cdot,t),t)$. Here we say that $\mathbf{w}$ is the *material velocity field* and write

$$\mathbf{w} = \mathbf{w}_\nu + \mathbf{w}_a \tag{2.2}$$

where $\mathbf{w}_\nu$ is the given normal velocity of the evolving hypersurface and $\mathbf{w}_a$ is a given tangential velocity field.

In the next two definitions, we suppose that u is a sufficiently smooth function defined on $\{\Gamma(t)\}_{t\in[0,T]}$ (see §2.4.1 later).

Definition 2.2.4 (Normal time derivative)**.** Suppose that the hypersurface $\{\Gamma(t)\}$ evolves with a normal velocity $\mathbf{w}_\nu$. The *normal time derivative* is defined by

$$\partial^\circ u := u_t + \nabla u \cdot \mathbf{w}_\nu.$$

Definition 2.2.5 (Material derivative)**.** Suppose that the hypersurface $\Gamma(t)$ evolves with a normal velocity $\mathbf{w}_\nu$. Given a tangential velocity field $\mathbf{w}_a$, with $\mathbf{w}$ as in (2.2), the *material derivative* is defined by

$$\partial^\bullet u := u_t + \nabla u \cdot \mathbf{w}. \tag{2.3}$$

We also write $\dot{u}$ for $\partial^\bullet u$. See [33, 35].

Remark 2.2.6 (Velocity fields)**.** It is useful to note that there are different notions of velocities for an evolving hypersurface.

- Suppose that the velocity $\mathbf{w}$ of an evolving compact hypersurface is purely tangential (so $\mathbf{w} \cdot \nu = 0$). In this case, material points on the initial surface get transported across the surface over time but *the surface remains the same.* One can see this for a sufficiently smooth initial surface Γ_0 by supposing that Γ_0 is the zero-level set of a function $\Psi\colon \mathbb{R}^{n+1} \to \mathbb{R}$:

  $$\Gamma_0 = \{x \in \mathbb{R}^{n+1} \mid \Psi(x) = 0\}.$$

  Let P be a material point on Γ_0 and $\gamma(t)$ denote the position of P at time t, with $\gamma(t) \in \Gamma(t)$. Then a purely tangential velocity means that $\nabla\Psi(\gamma(t)) \cdot \gamma'(t) = 0$, but this is precisely

  $$\frac{d}{dt}\Psi(\gamma(t)) = 0,$$

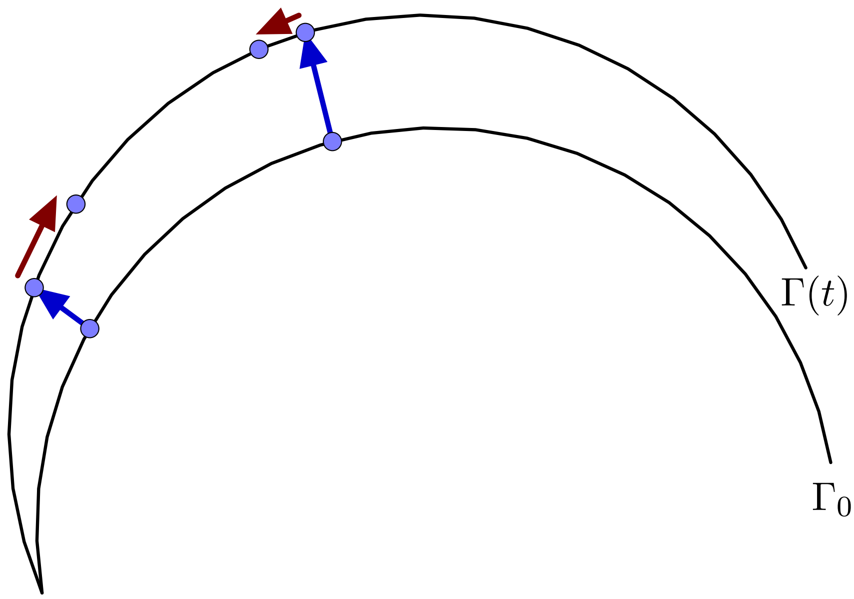

Figure 2.2.1: A sketch of the evolution of two material points on an evolving curve. The normal motion is given by the blue arrows and the tangential motion is given by the red arrows.

so the point persists in being a zero of the level set. Since P was arbitrary, we conclude that $\Gamma(t)$ coincides with Γ_0 for all $t \in [0, T]$, i.e.,

$$\Gamma(t) = \{x \in \mathbb{R}^{n+1} \mid \Psi(x) = 0\}.$$

- In applications, there may be a *physical velocity*

$$\mathbf{w}_\nu + \mathbf{w}_\tau,$$

where $\mathbf{w}_\nu$ is the normal component and $\mathbf{w}_\tau$ is the tangential component. The tangential velocity may be associated with the motion of physical material points and may be relevant to the mathematical models of processes on the surface.

- The velocity field (2.2) defines the path that points on the initial surface take with respect to the mapping F. In finite element analysis, it may be necessary to choose the tangential velocity $\mathbf{w}_a$ in an ALE approach so as to yield a shape-regular or adequately refined mesh. See [59] and [50, §5.7] for more details on this. One may wish to use this physical tangential velocity to define the map F. In writing down PDEs on evolving surfaces it is important to distinguish these notions.

- In certain situations, it can be useful to consider on an evolving surface a *boundary velocity* $\mathbf{w}_b$ which we can extend (arbitrarily) to the interior. In the case of flat hypersurfaces with $\mathbf{w}_\nu \equiv 0$ (this is the case when an evolving domain in $\mathbb{R}^n$ is viewed like in Remark 2.2.3), the conormal component of the arbitrary velocity must agree with the conormal component of the boundary velocity $\mathbf{w}_b$, otherwise the velocities map to two different surfaces.

2.3 The equations

We now state the equations we will study. Three of the problems are posed on evolving bounded open sets in $\mathbb{R}^n$. In this case, we shall denote by $\Omega(t)$ the evolving domain and $\Gamma(t)$ will denote the evolving compact hypersurface $\partial\Omega(t)$. In the equations given below, $\mathbf{w}$ is a velocity field which has a normal component $\mathbf{w}_\nu$ agreeing with the normal velocity of the evolving hypersurface or domain associated to the problem and an arbitrary tangential component $\mathbf{w}_a$.

Surface heat equation Suppose we have an evolving compact hypersurface $\Gamma(t)$ that evolves with normal velocity $\mathbf{w}_\nu$. Given a surface flux $\mathbf{q}$, we consider the conservation law

$$\frac{d}{dt}\int_{M(t)} u = -\int_{\partial M(t)} \mathbf{q}\cdot\mu$$

on an arbitrary portion $M(t) \subset \Gamma(t)$, where μ denotes the conormal on $\partial M(t)$. Without loss of generality we can assume that $\mathbf{q}$ is tangential. This conservation law implies the pointwise equation $u_t + \nabla u \cdot \mathbf{w}_\nu + u\nabla_\Gamma \cdot \mathbf{w}_\nu + \nabla_\Gamma \cdot \mathbf{q} = 0$. Assuming that the flux is a combination of a diffusive flux and an advective flux, so that $\mathbf{q} = -\nabla_\Gamma u + u\mathbf{b}_\tau$ where $\mathbf{b}_\tau$ is an advective tangential velocity field, we obtain $u_t + \nabla u \cdot \mathbf{w}_\nu + u\nabla_\Gamma \cdot \mathbf{w}_\nu - \Delta_\Gamma u + \nabla_\Gamma u \cdot \mathbf{b}_\tau + u\nabla_\Gamma \cdot \mathbf{b}_\tau = 0$. Setting $\mathbf{b} = \mathbf{w}_\nu + \mathbf{b}_\tau$, and recalling (2.3), we end up with the surface heat equation

$$\begin{aligned} \dot{u} - \Delta_\Gamma u + u\nabla_\Gamma \cdot \mathbf{b} + \nabla_\Gamma u \cdot (\mathbf{b} - \mathbf{w}) &= 0 \\ u(0) &= u_0 \end{aligned} \tag{2.4}$$

supplemented with an initial condition $u_0 \in L^2(\Gamma_0)$. Clearly, this surface heat equation is the archetypal example of a parabolic equation on a moving domain so fitting it into the framework is worthwhile. The heat equation on an evolving surface was first considered by Dziuk and Elliott in [48] where well-posedness (in slightly different function spaces) through a Galerkin method and finite element analysis was done. These types of reaction-diffusion equations have an extensive literature; let us name a few papers. See [9] for numerical analysis of such equations with a nonlinear source term, [59] for an ALE evolving surface finite element method, [125] for well-posedness and optimal control, and see also the introduction of Chapter 1 for many other references.

A bulk equation With $f(t)\colon \Omega(t) \to \mathbb{R}$ and $u_0\colon \Omega_0 \to \mathbb{R}$ given, consider the boundary value problem

$$\begin{aligned} \dot{u}(t) + (\mathbf{b}(t) - \mathbf{w}(t)) \cdot \nabla u(t) + u(t)\nabla \cdot \mathbf{b}(t) - D\Delta u(t) &= f(t) && \text{on } \Omega(t) \\ u(t, \cdot) &= 0 && \text{on } \Gamma(t) \\ u(0, \cdot) &= u_0(\cdot) && \text{on } \Omega_0 \end{aligned} \tag{2.5}$$

where $D > 0$ is a constant and the physical material velocity $\mathbf{b}(t)\colon \Omega(t) \to \mathbb{R}^n$ is sufficiently smooth with $\|\mathbf{b}(t)\|_{L^\infty(\Omega(t))} \leq C_1$ and $\|\nabla \cdot \mathbf{b}(t)\|_{L^\infty(\Omega(t))} \leq C_2$ for constants C_1 and C_2 uniform for all almost time. We refer the reader to [38] for a formulation of balance equations on moving time-dependent bulk domains. The problem (2.5) is a moving *hypersurface* version of a problem considered in [21] on a moving *domain*. This type of equation is a first approximation of Navier–Stokes equations describing fluid-structure interactions [21].

A coupled bulk-surface system In [55], the authors consider the well-posedness of an elliptic coupled bulk-surface system on a (static) domain; we now extend this to the parabolic case in a moving framework. These types of models arise in cell biology and in particular in cellular signalling and metabolism which can be mediated by membrane receptors in the interior that can diffuse on the boundary (the cell membrane). There may also be diffusion processes on the boundary coupled to diffusion processes in the interior [36]. These types of coupled bulk-surface problems are abundant in the mathematical biology literature [88, 37, 103] and indeed these applications are a rich source of interesting PDE problems for analysts. Recently, a parabolic coupled bulk-surface system arising from modelling receptor-ligand dynamics in cells was shown to be well posed in [56] (on a stationary domain). A number of free boundary problems were also derived as limits of certain parameters in the coupled system. There, the coupling between the interior and the boundary quantities is nonlinear. As a start, we will study a linear problem on an evolving domain.

Given $f(t) \colon \Omega(t) \to \mathbb{R}$, $g(t) \colon \Gamma(t) \to \mathbb{R}$, $u_0 \in H^1(\Omega_0)$ and $v_0 \in H^1(\Gamma_0)$, we want to find solutions $u(t) \colon \Omega(t) \to \mathbb{R}$ and $v(t) \colon \Gamma(t) \to \mathbb{R}$ of the coupled bulk-surface system

$$
\begin{aligned}
\dot{u} - \Delta_\Omega u + u \nabla_\Omega \cdot \mathbf{w} &= f && \text{on } \Omega(t) && (2.6)\\
\dot{v} - \Delta_\Gamma v + v \nabla_\Gamma \cdot \mathbf{w} + \nabla_\Omega u \cdot \nu &= g && \text{on } \Gamma(t) && (2.7)\\
\nabla_\Omega u \cdot \nu &= \beta v - \alpha u && \text{on } \Gamma(t) && (2.8)\\
u(0) &= u_0 && \text{on } \Omega_0 && (2.9)\\
v(0) &= v_0 && \text{on } \Gamma_0 && (2.10)
\end{aligned}
$$

where α, $\beta > 0$ are constants. One can think of u and v as being chemical species interacting through the Robin boundary condition (2.8). Note that we reused the notation u for denoting the trace of u. We use the physical material velocity to define the mapping F and assume there is just the one velocity field $\mathbf{w}$ which advects u within Ω and v on Γ.

A dynamic boundary problem for an elliptic equation Given $v_0 \in L^2(\Gamma_0)$ and $f(t) \in H^{-\frac{1}{2}}(\Gamma(t))$, we consider the problem of finding a function $v(t) \colon \Omega(t) \to \mathbb{R}$ such that, with $u(t) := v(t)|_{\Gamma(t)}$ denoting the trace,

$$
\begin{aligned}
\Delta v(t) &= 0 && \text{on } \Omega(t)\\
\dot{u}(t) + \frac{\partial v(t)}{\partial \nu(t)} + u(t) &= f(t) && \text{on } \Gamma(t) \qquad (2.11)\\
u(0) &= v_0 && \text{on } \Gamma_0
\end{aligned}
$$

holds in a weak sense. Here we assume that $\Gamma(t)$ evolves with the velocity $\mathbf{w}$ which we suppose is a normal velocity. This is a natural (linearised) extension to evolving domains of a problem considered by Lions in [84, §1.11.1]. One reason why problems with dynamic boundary conditions are interesting is because (by definition) the paraboic nature of the problem is found in the boundary of the domain and this leads to a more interesting functional setting. It will turn out that this problem can be formulated in the fractional Sobolev space $H^{\frac{1}{2}}(\Gamma(t))$, meaning that we need to check a number of technical assumptions on this kind of space in order to apply the abstract framework. Doing this work here becomes enormously useful when we study a fractional porous medium equation in Chapter 4.

With regards to other work, in [124] a heat equation with linear dynamical boundary conditions is treated on a bounded domain, where well- and ill-posedness results are given. In [126], the problem studied is a Laplace equation with zero Dirichlet and nonlinear dynamical boundary conditions on two disjoint parts of the boundary of the domain, and the author proves existence through the use of Dirichlet-to-Neumann maps like we also will do. The latter paper also contains many references for the curious reader. Both papers are in the setting of stationary domains, so our work, though linear, is new.

In order to formulate these equations in an appropriate weak sense and carry out the analysis, we will need Bochner-type function spaces for evolving hypersurfaces and the associated theory. This is done in the next section.

2.4 Function spaces on evolving hypersurfaces and domains

We now discuss evolving compact hypersurfaces (as defined in §2.2) and evolving domains in the context of the abstract framework presented in Chapter 1.

2.4.1 Evolving compact hypersurfaces

For each $t \in [0,T]$, let $\Gamma(t) \subset \mathbb{R}^{n+1}$ be a compact (i.e., no boundary) n-dimensional hypersurface of class C^2, and assume the existence of a flow $\Phi\colon [0,T]\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that for all $t \in [0,T]$, with $\Gamma_0 := \Gamma(0)$, the map $\Phi_t^0(\cdot) := \Phi(t,\cdot)\colon \Gamma_0 \to \Gamma(t)$ is a C^2-diffeomorphism that satisfies

$$\begin{aligned} \frac{d}{dt}\Phi_t^0(\cdot) &= \mathbf{w}(t,\Phi_t^0(\cdot)) \\ \Phi_0^0(\cdot) &= \mathrm{Id}(\cdot), \end{aligned} \tag{2.12}$$

where the map $\mathbf{w}\colon [0,T]\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a velocity field (with normal component agreeing with the normal velocity of $\Gamma(t)$), and we assume that it is C^2 and satisfies the uniform bound

$$|\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)| \leq C \qquad \text{for all } t \in [0,T].$$

A normal vector field on the hypersurface is denoted by $\nu\colon [0,T]\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. Let $V(t) = H^1(\Gamma(t))$ and $H(t) = L^2(\Gamma(t))$. We define the pullback operator by

$$\phi_{-t}v = v \circ \Phi_t^0.$$

By [125, Lemma 3.2], the map ϕ_{-t} is such that

$$\phi_{-t}\colon L^2(\Gamma(t)) \to L^2(\Gamma_0) \qquad \text{and} \qquad \phi_{-t}\colon H^1(\Gamma(t)) \to H^1(\Gamma_0)$$

are linear homeomorphisms with the constants of continuity not dependent on t. We denote by $\phi^*_{-t}\colon H^{-1}(\Gamma_0) \to H^{-1}(\Gamma(t))$ the dual operator. The maps $t \mapsto \|\phi_t u\|_{X(t)}$ (for $X = L^2$ and H^1) are continuous [125, Lemma 3.3], thus we have compatibility of the pairs $(H, \phi_{(\cdot)})$ and $(V, \phi_{(\cdot)}|_V)$, and the spaces $L^2_H = L^2_{L^2}$, $L^2_V = L^2_{H^1}$ and $L^2_{V^*} = L^2_{H^{-1}}$ are well-defined.

Let us now work out a formula for the strong material derivative. Note that, by the smoothness of $\Gamma(t)$, any function $u\colon \Gamma(t) \to \mathbb{R}$ can be extended to a neighbourhood of the space time surface $\cup_{t\in[0,T]}\Gamma(t) \times \{t\}$ in $\mathbb{R}^{n+2}$ in which ∇u and u_t for the extension are well-defined (see for example [50, §2.2]). The derivative of the pullback of a function $u \in C^1_V$ is

$$\begin{aligned}\frac{d}{dt}\phi_{-t}u(t) = \frac{d}{dt}u(t, \Phi^0_t(y)) &= u_t(t, \Phi^0_t(y)) + \nabla u|_{(t,\Phi^0_t(y))} \cdot \mathbf{w}(t, \Phi^0_t(y))\\ &= \phi_{-t}u_t(t, y) + \phi_{-t}(\nabla u(t, y)) \cdot \phi_{-t}(\mathbf{w}(t, y)), \quad y \in \Gamma_0\end{aligned}$$

giving $\dot{u}(t, x) = u_t(t, x) + \nabla u(t, x) \cdot \mathbf{w}(t, x)$ for $x \in \Gamma(t)$. The expression on the right hand side is independent of the extension. It is clear that our definition of the strong material derivative coincides with the well-established definition (2.3).

We denote by J^0_t the change of area element when transforming from Γ_0 to $\Gamma(t)$, i.e., for any integrable function $\zeta\colon \Gamma(t) \to \mathbb{R}$

$$\int_{\Gamma(t)} \zeta = \int_{\Gamma_0} (\zeta \circ \Phi^0_t)J^0_t = \int_{\Gamma_0} \phi_{-t}\zeta J^0_t.$$

Using the transport identity

$$\frac{d}{dt}\int_{G(t)} \zeta(t)\Big|_t = \int_{G(t)} \dot{\zeta}(t) + \zeta(t)\nabla_{G(t)} \cdot \mathbf{w}(t)$$

on any portion $G \subset \Gamma$ with points that move with the velocity field $\mathbf{w}$ (for instance, see [48]) one can easily show that

$$\frac{d}{dt}J^0_t = \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t))J^0_t. \tag{2.13}$$

The field J_t^0 is uniformly bounded by positive constants

$$\frac{1}{C_J} \leq J_t^0(z) \leq C_J \qquad \text{for all } z \in \Gamma_0 \text{ and for all } t \in [0, T].$$

The $L^2(\Gamma(t))$ inner product is

$$(u, v)_{L^2(\Gamma(t))} = \int_{\Gamma(t)} uv = \int_{\Gamma_0} \phi_{-t} u \phi_{-t} v J_t^0.$$

The bilinear form $\hat{b}(t; \cdot, \cdot)\colon H_0 \times H_0 \to \mathbb{R}$ (defined by $(u, v)_{H(t)} = \hat{b}(\phi_{-t}v, \phi_{-t}v)$) is

$$\hat{b}(t; u_0, v_0) = \int_{\Gamma_0} u_0 v_0 J_t^0,$$

so the action of the operator $T_t\colon H_0 \to H_0$ (see Definition 1.2.33 and Theorem 1.2.46) is just pointwise multiplication:

$$T_t u_0 = J_t^0 u_0.$$

We see that the function θ from Assumptions 1.2.35 is

$$\begin{aligned}\theta(t, u_0) &= \frac{d}{dt} \|\phi_t u_0\|^2_{L^2(\Gamma(t))} = \frac{d}{dt} \int_{\Gamma_0} u_0^2 J_t^0 = \int_{\Gamma_0} u_0^2 \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)) J_t^0 \\ &= \int_{\Gamma(t)} (\phi_t u_0)^2 \nabla_\Gamma \cdot \mathbf{w}(t),\end{aligned}$$

where the cancellation of the Jacobian terms in the last equality is due to the inverse function theorem. Now, $v \mapsto \theta(t, v)$ is continuous because if $v_n \to v$ in $L^2(\Gamma_0)$, then $v_n^2 \to v^2$ in $L^1(\Gamma_0)$ and so

$$|\theta(t, v_n) - \theta(t, v)| \leq \int_{\Gamma_0} |v_n^2 - v^2| |\phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)) J_t^0| \leq C \left\|v_n^2 - v^2\right\|_{L^1(\Gamma_0)} \to 0.$$

Finally,

$$\begin{aligned}|\theta(t, u_0 + v_0) - \theta(t, u_0 - v_0)| &= \left|4 \int_{\Gamma(t)} \phi_t u_0 \phi_t v_0 \nabla_{\Gamma(t)} \cdot \mathbf{w}(t)\right| \\ &\leq C \|u_0\|_{L^2(\Gamma_0)} \|v_0\|_{L^2(\Gamma_0)}.\end{aligned}$$

So we have checked Assumptions 1.2.35. Now if u_0, $v_0 \in L^2(\Gamma_0)$,

$$\hat{\lambda}(t; u_0, v_0) = \frac{\partial}{\partial t} \hat{b}(t; u_0, v_0) = \int_{\Gamma_0} u_0 v_0 \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}) J_t^0,$$

thus the bilinear form $\lambda(t;\cdot,\cdot)$ of Definition 1.2.36 is

$$\lambda(t;u,v) = \int_{\Gamma_0} \phi_{-t}u\phi_{-t}v\phi_{-t}(\nabla_{\Gamma(t)}\cdot\mathbf{w})J_t^0 = \int_{\Gamma(t)} uv\nabla_{\Gamma(t)}\cdot\mathbf{w},$$

which, as claimed, is measurable in t and bounded on $H(t)\times H(t)$. So then $u \in L^2_V$ has a weak material derivative $\dot{u} \in L^2_{V^*}$ if and only if

$$\int_0^T \langle \dot{u}(t), \eta(t)\rangle_{V^*(t),V(t)} = -\int_0^T \int_{\Gamma(t)} u(t)\dot{\eta}(t) - \int_0^T \int_{\Gamma(t)} u(t)\eta(t)\nabla_{\Gamma(t)}\cdot\mathbf{w}(t)$$

holds for all $\eta \in \mathcal{D}_V(0,T)$ (cf. [125, 98]).

Finally, [125, Lemma 3.7] proves that $T_{(\cdot)}u(\cdot) \in \mathcal{W}(V_0,V_0^*)$ if and only if $u \in \mathcal{W}(V_0,V_0^*)$, due to the fact that both $J^0_{(\cdot)}$ and its reciprocal $1/J^0_{(\cdot)}$ are in $C^1([0,T]\times\Gamma_0)$. To see that the evolving space equivalence (Assumption 1.2.44) holds, take $u \in \mathcal{W}(V_0,V_0^*)$ and obtain by the product rule and (2.13) the identity

$$(J_t^0u(t))' = J_t^0u'(t) + \phi_{-t}(\nabla_{\Gamma(t)}\cdot\mathbf{w})J_t^0u(t).$$

Therefore, the maps $\hat{S}(t)$ and $\hat{D}(t)$ from Theorem 1.2.46 are $\hat{S}(t)u'(t) = J_t^0u'(t)$ and $\hat{D}(t) \equiv 0$. It follows by the smoothness of Φ_t^0 and J_t^0 that $\hat{S}(\cdot)u'(\cdot) \in L^2(0,T;V_0^*)$. By Theorem 1.2.46, we have that the space $W(V,V^*) = \{u \in L^2_{H^1} \mid \dot{u} \in L^2_{H^{-1}}\}$ is indeed isomorphic to $\mathcal{W}(V_0,V_0^*)$ and there is an equivalence of norms between

$$\|u\|_{W(V,V^*)} \qquad \text{and} \qquad \left\|\phi_{-(\cdot)}u(\cdot)\right\|_{\mathcal{W}(V_0,V_0^*)}.$$

See also [125, Lemma 3.9]. It is easy to see that $W(V,H)$ and $\mathcal{W}(V_0,H_0)$ are also equivalent.

2.4.2 Evolving domains

We discuss here what is common to the three examples on evolving domains and leave the specifics and peculiarities to be detailed on a case-by-case basis as required.

For each $t \in [0,T]$, let $\Omega(t) \subset \mathbb{R}^n$ be a bounded open and connected domain of class C^2 with boundary $\Gamma(t)$. It is possible to view $\Omega(t)$ as an evolving flat hypersurface in $\mathbb{R}^{n+1}$ (see Remark 2.2.3), though we choose not to follow this approach. The boundary $\Gamma(t)$ is an evolving compact $(n-1)$-dimensional hypersurface in $\mathbb{R}^n$. We denote $\Omega_0 := \Omega(0)$ and $\Gamma_0 := \Gamma(0)$. For each $t \in [0,T]$, we assume the existence of a map $\Phi_t^0\colon \overline{\Omega}_0 \to \overline{\Omega(t)}$ such that $\Phi_t^0(\Omega_0) = \Omega(t)$, $\Phi_t^0(\Gamma_0) = \Gamma(t)$,

$$\Phi_t^0\colon \Omega_0 \to \Omega(t) \text{ is a } C^2\text{-diffeomorphism} \qquad \text{and} \qquad \Phi_{(\cdot)}^0 \in C^2([0,T] \times \overline{\Omega_0}).$$

We assume that Φ_t^0 satisfies the ODE (2.12) on $\overline{\Omega_0}$ for a C^2 velocity $\mathbf{w}$ (with the normal part of $\mathbf{w}$ agreeing with the normal velocity of the domain) with $|\nabla\cdot\mathbf{w}(t)|$ and $|\nabla_{\Gamma(t)}\cdot\mathbf{w}(t)|$ both bounded above uniformly in t, like before. We write $\Phi_0^t := (\Phi_t^0)^{-1}$.

Definition 2.4.1. For functions $u\colon \Omega_0 \to \mathbb{R}$ and $v\colon \Gamma_0 \to \mathbb{R}$, define the restrictions

$$\phi_{\Omega,t}u = u \circ \Phi_0^t|_{\Omega_0} \qquad \text{and} \qquad \phi_{\Gamma,t}v = v \circ \Phi_0^t|_{\Gamma_0}.$$

We find that

$$\phi_{\Omega,t}\colon H^1(\Omega_0) \to H^1(\Omega(t)) \qquad \text{and} \qquad \phi_{\Omega,t}\colon L^2(\Omega_0) \to L^2(\Omega(t))$$

are linear homeomorphisms with the constants of continuity not depending on t (we can either adapt the proofs in [125] or use Problem 1.3.1 in [94]). One of the most important terms in the solution space regime is the Jacobian $J_{\Omega,(\cdot)}^0 := \det\mathbf{D}\Phi_{(\cdot)}^0 \in C^1([0,T] \times \Omega_0)$; one can show that it satisfies much of the same properties (see [21] for this) as the Jacobian term did in §2.4.1 for the case of compact hypersurfaces. Hence it is straightforward to adapt the proofs for the case of a domain with boundary to yield the fulfilment of the evolving space equivalence Assumption 1.2.50 between $\mathcal{W}(H^1(\Omega_0), (H^1(\Omega_0))^*)$ and $W(H^1_\Omega, (H^1_\Omega)^*)$, and $\mathcal{W}(H^1(\Omega_0), L^2(\Omega_0))$ and $W(H^1_\Omega, L^2_\Omega)$.

Furthermore, assuming

$$\Phi_t^0\colon \Gamma_0 \to \Gamma(t) \text{ is a } C^2\text{-diffeomorphism},$$

since the boundary $\Gamma(t)$ is a C^2 hypersurface, it satisfies the assumptions in §2.4.1 and so it follows that the maps

$$\phi_{\Gamma,t}\colon H^1(\Gamma_0) \to H^1(\Gamma(t)) \qquad \text{and} \qquad \phi_{\Gamma,t}\colon L^2(\Gamma_0) \to L^2(\Gamma(t))$$

are also linear homeomorphisms with the constants of continuity not depending on t. The trace map $\tau_t\colon H^1(\Omega(t)) \to L^2(\Gamma(t))$ (see [127, §I.8, Theorem 8.7]) will play a prominent role. We need the following lemma to show that the constant in the trace inequality is uniform in time.

Lemma 2.4.2. For all $w \in H^1(\Omega_0)$, the equality $\tau_t(\phi_{\Omega,t}w) = \phi_{\Gamma,t}(\tau_0 w)$ holds in

$L^2(\Gamma(t))$.

Proof. This is because $\tau_t(\phi_{\Omega,t}w_n) = \phi_{\Gamma,t}(\tau_0 w_n)$ holds for all $w_n \in C^1(\overline{\Omega_0})$ (one can see this identity by using the fact that the same formula defines $\phi_{\Omega,t}$ and $\phi_{\Gamma,t}$ and that Φ_0^t maps boundary to boundary), in particular, it holds for $w_n \in C^1(\overline{\Omega_0}) \cap H^1(\Omega_0)$ such that $w_n \to w$ in $H^1(\Omega_0)$. Then by continuity of the various maps, we can pass to the limit and obtain the identity. $\square$

Now let $u \in H^1(\Omega_0)$. Using Lemma 2.4.2 and the properties of the maps $\phi_{\Gamma,t}$ and $\phi_{\Omega,t}$, we obtain

$$\|\tau_0 u\|_{L^2(\Gamma_0)} \geq C_1 \|\phi_{\Gamma,t}(\tau_0 u)\|_{L^2(\Gamma(t))} = C_1 \|\tau_t(\phi_{\Omega,t}u)\|_{L^2(\Gamma(t))}$$

and

$$\|u\|_{H^1(\Omega_0)} \leq C_2 \|\phi_{\Omega,t}u\|_{H^1(\Omega(t))},$$

and these inequalities together with the trace inequality on Ω_0 imply the existence of C_T such that

$$\|\tau_t u\|_{L^2(\Gamma(t))} \leq C_T \|u\|_{H^1(\Omega(t))} \quad \forall u \in H^1(\Omega(t)), \forall t \in [0,T]. \tag{2.14}$$

Remark 2.4.3. Observe that the velocity field $\mathbf{w}$ may have no physical or actual relevance to a particular problem posed on an evolving hypersurface apart from having the normal component of $\mathbf{w}$ agreeing with the normal velocity of the hypersurface (or domain). The tangential component of $\mathbf{w}$ can be chosen arbitrarily — a fact which can exploited for numerics, as mentioned before in Remark 2.2.6. On the other hand, $\mathbf{w}$ plays an indispensable role in the definition of the function spaces in which we look for solutions.

2.5 Weak formulation and well-posedness

We are now in a position to prove the well-posedness of the equations in §2.3 in a weak sense.

2.5.1 The surface advection-diffusion equation (2.4)

Let us assume for simplicity that $\mathbf{b} = \mathbf{w}$ in (2.4); that is, the physical velocity agrees with the velocity of the parametrisation. Let us suppose that $\Gamma(t)$ possesses the properties in §2.4.1. Availing ourselves of the framework in §2.4.1, the weak formulation of (2.4) asks to find $u \in W(V, V^*)$ such that

$$\int_0^T \langle \dot{u}(t), v(t) \rangle_{H^{-1}(\Gamma(t)),H^1(\Gamma(t))} + \int_{\Gamma(t)} \nabla_\Gamma u(t) \cdot \nabla_\Gamma v(t) + \int_{\Gamma(t)} u(t)v(t)\nabla_\Gamma \cdot \mathbf{w}(t) = 0$$

holds for all $v \in L^2_V$. Here,

$$a(t; u, v) = \int_{\Gamma(t)} \nabla_\Gamma u \cdot \nabla_\Gamma v$$

which clearly satisfies the assumptions listed in Assumptions 1.3.3. Applying Theorem 1.4.1, we obtain a unique solution $u \in W(V, V^*)$. If instead we ask for $\dot{u} \in L^2_H$, in addition to requiring $u_0 \in H^1(\Gamma_0)$, we need to check Assumptions 1.4.2 and 1.4.6; the former follows since for example we can take χ_j^0 to be the eigenfunctions of the Laplacian (see Remark 1.4.3). We take $a_s \equiv a$ as defined above and set $a_n \equiv 0$. Most of the remaining assumptions are easy to check. For (A3), we see from [48, Lemma 2.2] that for $\eta \in C^\infty_V$, the pointwise derivative

$$\begin{aligned}\frac{d}{dt}&\int_{\Gamma(t)} |\nabla_\Gamma \eta(t)|^2 \\ &= \int_{\Gamma(t)} (2\nabla_\Gamma \eta(t) \cdot \nabla_\Gamma \dot{\eta}(t) - 2\nabla_\Gamma \eta(t)(\mathbf{D}_\Gamma \mathbf{w}(t))\nabla_\Gamma \eta(t) + |\nabla_\Gamma \eta(t)|^2 \nabla_\Gamma \cdot \mathbf{w}(t))\end{aligned} \tag{2.15}$$

holds everywhere with $(\mathbf{D}_\Gamma \mathbf{w}(t))_{ij} := \underline{D}_j \mathbf{w}^i(t)$. Since the right hand side of the above expression is in $L^1(0, T)$, we have that the derivative is in fact a weak derivative. By a density argument, we find that the formula above holds in the weak sense also for $\eta \in \tilde{C}^1_V$. Since the right hand side and the term being differentiated on the left hand side are in $L^1(0, T)$, it follows that $t \mapsto \int_{\Gamma(t)} |\nabla_\Gamma \eta(t)|^2$ has an absolutely continuous representative with the pointwise a.e. derivative as above, giving (A7). It is easy to see that

$$r(t; \eta) = \int_{\Gamma(t)} (-2\nabla_\Gamma \eta(\mathbf{D}_\Gamma \mathbf{w}(t))\nabla_\Gamma \eta + |\nabla_\Gamma \eta|^2 \nabla_\Gamma \cdot \mathbf{w}(t))$$

satisfies (A8). Finally, an application of Theorem 1.4.8 shows that $u \in W(V, H)$.

Remark 2.5.1. We mentioned in Remark 2.2.6 that if $\mathbf{w}$ is purely tangential, the surface does not evolve. However, even in this situation, it can still be useful to think of spaces of functions on $\Gamma(t) \equiv \Gamma_0$ as $H(t)$ and $V(t)$ (i.e., still parametrised by $t \in [0, T]$). Consider the surface heat equation

$$\dot{u} - \Delta_\Gamma u + u \nabla_\Gamma \cdot \mathbf{w} = f.$$

If $\mathbf{w}(t, \cdot)$ is a tangential velocity field, then this equation corresponds to

$$u_t - \Delta_\Gamma u + u \nabla_\Gamma \cdot \mathbf{w} + \mathbf{w} \cdot \nabla_\Gamma u = f,$$

which could be advection-dominated (if $\mathbf{w}$ is sufficiently large) and potentially problematic for numerical computations. The first formulation, in which we make use of $H(t)$ and $V(t)$ for each $t \in [0, T]$, avoids this issue.

2.5.2 The bulk equation (2.5)

Here, we use the notations and results of §2.4.2. Observe that the velocity field $\mathbf{w}$ does not appear in the physical equation (2.5); $\mathbf{w}$ is an extension to the interior (or bulk) of the boundary velocity, and the normal component of this boundary velocity must agree with the normal velocity of $\Omega(t)$. For example, if the normal velocity of $\Omega(t)$ were $\mathbf{b} \cdot \nu$ then $\mathbf{w}$ can be taken to be an extension of $\mathbf{b} \cdot \nu$. In this sense, $\mathbf{w}$ is not relevant to the physical problem but it is essential to the functional setting we have built up (see Remark 2.4.3). Let $V(t) = H_0^1(\Omega(t))$ and $H(t) = L^2(\Omega(t))$. With ϕ_t referring to the map $\phi_{\Omega,t}$ from Definition 2.4.1, it follows from §2.4.2 that $(H, \phi_{(\cdot)})$ and $(V, \phi_{(\cdot)}|_V)$ are compatible and that there is an evolving space equivalence between $\mathcal{W}(V_0, V_0^*)$ and $W(V, V^*)$. For convenience, set $\mathbf{p} := \mathbf{b} - \mathbf{w}$. Our weak formulation is: with $f \in L^2_H$ and $u_0 \in V_0$, find $u \in W(V, H)$ such that

$$\begin{aligned}\int_0^T \int_{\Omega(t)} (\dot{u}(t)v(t) + \mathbf{p}(t) \cdot \nabla u(t)v(t) + \nabla \cdot \mathbf{b}(t)u(t)v(t) + D\nabla u(t) \cdot \nabla v(t)) \\ = \int_0^T \int_{\Omega(t)} f(t)v(t) \\ u(0) = u_0\end{aligned}$$

holds for all $v \in L^2_V$. Now, Assumption 1.4.2 holds just like in the previous example. We need to check Assumptions 1.3.3 and 1.4.6. We have

$$a(t; u, v) = \int_{\Omega(t)} \mathbf{p}(t) \cdot \nabla uv + (\nabla \cdot \mathbf{b}(t))uv + D\nabla u \cdot \nabla v$$

with

$$a_s(t;u,v) = \int_{\Omega(t)} D\nabla u \cdot \nabla v \;\text{ and }\; a_n(t;u,v) = \int_{\Omega(t)} ((\nabla \cdot \mathbf{b}(t))u + \mathbf{p}(t) \cdot \nabla u)v.$$

The boundedness of $a(t;\cdot,\cdot)$ is easy, while coercivity can be shown by the use of Young's equality with ϵ:

$$\begin{aligned} a(t;v,v) &\geq D \|\nabla v\|^2_{L^2(\Omega(t))} - \frac{C}{2D} \left\|\mathbf{p}^2(t)\right\|_{L^\infty(\Omega(t))} \|v\|^2_{L^2(\Omega(t))} - \frac{D}{2} \|\nabla v\|^2_{L^2(\Omega(t))} \\ &\quad - \|\nabla \cdot \mathbf{b}(t)\|_{L^\infty(\Omega(t))} \|v\|^2_{L^2(\Omega(t))} \\ &= -\left(\frac{C}{2D} \left\|\mathbf{p}^2(t)\right\|_{L^\infty(\Omega(t))} + \|\nabla \cdot \mathbf{b}(t)\|_{L^\infty(\Omega(t))}\right) \|v\|^2_{L^2(\Omega(t))} \\ &\quad + \frac{D}{2} \|\nabla v\|^2_{L^2(\Omega(t))}. \end{aligned}$$

Coming to the term $a_s(t;\cdot,\cdot)$; firstly, positivity and boundedness are obvious, and absolute continuity and a.e. differentiability are the same as for the bilinear form $a(t;\cdot,\cdot)$ in the previous example:

$$\frac{d}{dt} a_s(t;\eta(t),\eta(t)) = 2a_s(t;\dot{\eta}(t),\eta(t)) + r(t;\eta(t))$$

for $\eta \in \tilde{C}^1_V$, where

$$r(t;\eta(t)) = D \int_{\Omega(t)} (-2\nabla\eta(t)(\mathbf{D}\mathbf{w}(t))\nabla\eta(t) + |\nabla\eta(t)|^2 \nabla \cdot \mathbf{w}(t))$$

which is obviously bounded. Finally, the uniform bound on $a_n(t;\cdot,\cdot)\colon V(t) \times H(t) \to \mathbb{R}$ follows by the assumptions made on $\mathbf{b}$ in §2.3. With all the assumptions checked, we apply Theorem 1.4.8 and find a unique solution $u \in W(V,H)$.

2.5.3 The coupled bulk-surface system (2.6)–(2.10)

We are again going to use the framework of §2.4.2. The setting up of the function spaces is slightly more involved now.

Function spaces

Define the product Hilbert spaces

$$V(t) = H^1(\Omega(t)) \times H^1(\Gamma(t)) \qquad \text{and} \qquad H(t) = L^2(\Omega(t)) \times L^2(\Gamma(t))$$

which we equip with the inner products

$$
\begin{aligned}
((\omega_1, \gamma_1), (\omega_2, \gamma_2))_{H(t)} &= (\omega_1, \omega_2)_{L^2(\Omega(t))} + (\gamma_1, \gamma_2)_{L^2(\Gamma(t))} \\
((\omega_1, \gamma_1), (\omega_2, \gamma_2))_{V(t)} &= (\omega_1, \omega_2)_{H^1(\Omega(t))} + (\gamma_1, \gamma_2)_{H^1(\Gamma(t))}.
\end{aligned}
$$

Clearly $V(t) \subset H(t)$ is continuous and dense and both spaces are separable. The dual space of $V(t)$ is $V^*(t) = (H^1(\Omega(t)))^* \times H^{-1}(\Gamma(t))$ and the duality pairing is

$$
\langle (f_\omega, f_\gamma), (u_\omega, u_\gamma) \rangle_{V^*(t), V(t)} = \langle f_\omega, u_\omega \rangle_{(H^1(\Omega(t)))^*, H^1(\Omega(t))} + \langle f_\gamma, u_\gamma \rangle_{H^{-1}(\Gamma(t)), H^1(\Gamma(t))}.
$$

Define the map $\phi_t \colon H_0 \to H(t)$ by

$$
\phi_t((\omega, \gamma)) = (\phi_{\Omega,t}\omega, \phi_{\Gamma,t}\gamma)
$$

where $\phi_{\Omega,t}$ and $\phi_{\Gamma,t}$ are as defined previously. From §2.4.1 and §2.4.2, we find that $(H, \phi_{(\cdot)})$ and $(V, \phi_{(\cdot)}|_V)$ are compatible, and we have the evolving space equivalence between $\mathcal{W}(V_0, V_0^*)$ and $W(V, V^*)$.

To define the weak material derivative, note that because the inner product on $H(t)$ is a sum of the L^2 inner products on $\Omega(t)$ and $\Gamma(t)$, it follows that the bilinear form $\lambda(t; \cdot, \cdot)$ is

$$
\lambda(t; (\omega_1, \gamma_1), (\omega_2, \gamma_2)) = \lambda_\Omega(t; \omega_1, \omega_2) + \lambda_\Gamma(t; \gamma_1, \gamma_2)
$$

with

$$
\lambda_\Omega(t; \omega_1, \omega_2) = \int_{\Omega(t)} \omega_1 \omega_2 \nabla_\Omega \cdot \mathbf{w}(t) \quad \text{and} \quad \lambda_\Gamma(t; \gamma_1, \gamma_2) = \int_{\Gamma(t)} \gamma_1 \gamma_2 \nabla_\Gamma \cdot \mathbf{w}(t)
$$

being the bilinear forms associated with the material derivatives of the constituent spaces of the product space.

Weak formulation and well-posedness

To obtain the weak form, we let $(\omega, \gamma) \in L^2_V$ and take the inner product of (2.6) with ω and the inner product of (2.7) with γ:

$$
\int_{\Omega(t)} \dot{u}\omega + \int_{\Omega(t)} \nabla_\Omega u \cdot \nabla_\Omega \omega - \int_{\Gamma(t)} \omega \nabla_\Omega u \cdot \nu + \int_{\Omega(t)} u\omega \nabla_\Omega \cdot \mathbf{w} = \int_{\Omega(t)} f\omega \tag{2.16}
$$

$$
\int_{\Gamma(t)} \dot{v}\gamma + \int_{\Gamma(t)} \nabla_\Gamma v \cdot \nabla_\Gamma \gamma + \int_{\Gamma(t)} v\gamma \nabla_\Gamma \cdot \mathbf{w} + \int_{\Gamma(t)} \gamma \nabla_\Omega u \cdot \nu = \int_{\Gamma(t)} g\gamma. \tag{2.17}
$$

Multiplying (2.16) by α and (2.17) by β, taking the sum and substituting the boundary condition (2.8), we end up with

$$\alpha \int_{\Omega(t)} \dot{u}\omega + \beta \int_{\Gamma(t)} \dot{v}\gamma + \alpha \int_{\Omega(t)} \nabla_\Omega u \cdot \nabla_\Omega \omega + \beta \int_{\Gamma(t)} \nabla_\Gamma v \cdot \nabla_\Gamma \gamma + \alpha \int_{\Omega(t)} u\omega \nabla_\Omega \cdot \mathbf{w}$$
$$+ \beta \int_{\Gamma(t)} v\gamma \nabla_\Gamma \cdot \mathbf{w} + \int_{\Gamma(t)} (\beta v - \alpha u)(\beta\gamma - \alpha\omega) = \alpha \int_{\Omega(t)} f\omega + \beta \int_{\Gamma(t)} g\gamma.$$

Defining the bilinear forms

$$l(t; (\dot{u}, \dot{v}), (\omega, \gamma)) = \alpha \langle \dot{u}, \omega \rangle_{(H^1(\Omega(t)))^*, H^1(\Omega(t))} + \beta \langle \dot{v}, \gamma \rangle_{H^{-1}(\Gamma(t)), H^1(\Gamma(t))}$$
$$a(t; (u, v), (\omega, \gamma)) = \alpha \int_{\Omega(t)} \nabla_\Omega u \cdot \nabla_\Omega \omega + \int_{\Gamma(t)} \beta \nabla_\Gamma v \cdot \nabla_\Gamma \gamma + (\beta v - \alpha u)(\beta\gamma - \alpha\omega),$$

our weak formulation reads: given $(f, g) \in L^2_H$ and $(u_0, v_0) \in V_0$, find $(u, v) \in W(V, H)$ such that

$$\begin{aligned} \int_0^T (l(t; (\dot{u}, \dot{v}), (\omega, \gamma)) + a(t; (u, v), (\omega, \gamma)) + \lambda(t; (u, v), (\omega, \gamma))) \\ = \int_0^T ((\alpha f, \alpha g), (\omega, \gamma))_{H(t)} \\ (u(0), v(0)) = (u_0, v_0) \end{aligned} \qquad (\mathbf{P_{bs}})$$

for all $(\omega, \gamma) \in L^2_V$. Note that Assumption 1.4.2 holds due to the compactness of $V_0 \subset H_0$. Let us now check Assumptions 1.3.2.

Assumptions (L1)–(L8) We can write

$$l(t; (\dot{u}, \dot{v}), (\omega, \gamma)) = \langle L(t)(\dot{u}, \dot{v}), (\omega, \gamma) \rangle_{V^*(t), V(t)} = \langle (\alpha\dot{u}, \beta\dot{v}), (\omega, \gamma) \rangle_{V^*(t), V(t)},$$

i.e., $L(\dot{u}, \dot{v})$ is the functional $\int_0^T \langle (\alpha\dot{u}(t), \beta\dot{v}(t)), (\cdot)(t) \rangle_{V^*(t), V(t)}$, which obviously satisfies (L1). We see that $L\colon L^2_H \to L^2_H$, and when $(\dot{u}, \dot{v}) \in H(t)$,

$$\langle L(t)(\dot{u}, \dot{v}), (\omega, \gamma) \rangle = ((\alpha\dot{u}, \beta\dot{v}), (\omega, \gamma))_{H(t)},$$

so indeed $L(t)|_{H(t)}$ has range in $H(t)$ and $L(t)|_{V(t)}$ has range in $V(t)$. Assumptions (L2)–(L5) are immediate, and (L6) also follows easily. For (L7) and (L8), note that the map $\dot{L} \equiv 0$.

We also need to check Assumptions 1.3.3 and 1.4.6 on the bilinear form $a(t; \cdot, \cdot)$. Set $\mathbf{v_i} = (\omega_i, \gamma_i)$ for $i = 1, 2$. Coercivity of $a(t; \cdot, \cdot)$ (assumption (A1)) is achieved with no great difficulty (one uses the L^∞ bound on $\mathbf{w} \cdot \mu$, the trace inequality and Young's inequality with ϵ).

Assumption (A2) For boundedness of $a(t;\cdot,\cdot)$, we start with

$$|a(t;\mathbf{v_1},\mathbf{v_2})| \leq C \|\mathbf{v_1}\|_{V(t)} \|\mathbf{v_2}\|_{V(t)} + \int_{\Gamma(t)} |\beta^2\gamma_1\gamma_2 + \alpha^2\omega_1\omega_2 - \alpha\beta(\omega_1\gamma_2 + \gamma_1\omega_2)|. \tag{2.18}$$

The trace inequality (2.14) allows us to estimate the last term of (2.18) as follows:

$$\begin{aligned}
&\int_{\Gamma(t)} |\beta^2\gamma_1\gamma_2 + \alpha^2\omega_1\omega_2 - \alpha\beta(\omega_1\gamma_2 + \gamma_1\omega_2)| \\
&\quad \leq \beta^2 \|\gamma_1\|_{L^2(\Gamma(t))} \|\gamma_2\|_{L^2(\Gamma(t))} + \alpha^2 C_T^2 \|\omega_1\|_{H^1(\Omega(t))} \|\omega_2\|_{H^1(\Omega(t))} \\
&\qquad + \alpha\beta C_T \left(\|\omega_1\|_{H^1(\Omega(t))} \|\gamma_2\|_{L^2(\Gamma(t))} + \|\gamma_1\|_{L^2(\Gamma(t))} \|\omega_2\|_{H^1(\Omega(t))} \right) \\
&\quad \leq C \|(\omega_1,\gamma_1)\|_{V(t)} \|(\omega_2,\gamma_2)\|_{V(t)} = C \|\mathbf{v_1}\|_{V(t)} \|\mathbf{v_2}\|_{V(t)}.
\end{aligned}$$

Assumptions (A7) **and** (A8) We do not require the splitting of $a(t;\cdot,\cdot)$ into a differentiable and non-differentiable part since $a(t;\cdot,\cdot)$ is differentiable as shown below (the absolute continuity follows like before). In view of this and Remark 1.4.7, we still need to check (A7) and (A8). Let us define

$$a_\Omega(t;\omega_1,\omega_2) = \alpha \int_{\Omega(t)} \nabla_\Omega\omega_1 \cdot \nabla_\Omega\omega_2 \qquad \text{and} \qquad a_\Gamma(t;\gamma_1,\gamma_2) = \beta \int_{\Gamma(t)} \nabla_\Gamma\gamma_1 \cdot \nabla_\Gamma\gamma_2,$$

so that

$$a(t;(\omega_1,\gamma_1),(\omega_2,\gamma_2)) = a_\Omega(t;\omega_1,\omega_2) + a_\Gamma(t;\gamma_1,\gamma_2) + \int_{\Gamma(t)} (\beta\gamma_1 - \alpha\omega_1)(\beta\gamma_2 - \alpha\omega_2)$$

Taking $\mathbf{v_1} \in \tilde{C}^1_V$, we differentiate:

$$\begin{aligned}
\frac{d}{dt}a(t;\mathbf{v_1},\mathbf{v_1}) &= 2a_\Omega(t;\dot{\omega}_1,\omega_1) + r_\Omega(t;\omega_1) + 2a_\Gamma(t;\dot{\gamma}_1,\gamma_1) + r_\Gamma(t;\gamma_1) \\
&\quad + 2(\beta\dot{\gamma}_1 - \alpha\dot{\omega}_1, \beta\gamma_1 - \alpha\omega_1)_{L^2(\Gamma(t))} + \lambda_\Gamma(t;\beta\gamma_1 - \alpha\omega_1, \beta\gamma_1 - \alpha\omega_1) \\
&= 2a(t;(\dot{\omega}_1,\dot{\gamma}_1),(\omega_1,\gamma_1)) + r(t;(\omega_1,\gamma_1)) \\
&= 2a(t;\dot{\mathbf{v_1}},\mathbf{v_1}) + r(t;\mathbf{v_1}).
\end{aligned}$$

Here, we defined

$$r(t;(\omega_1,\gamma_1)) = r_\Omega(t;\omega_1) + r_\Gamma(t;\gamma_1) + \lambda_\Gamma(t;\beta\gamma_1-\alpha\omega_1,\beta\gamma_1-\alpha\omega_1)$$

where r_Ω and r_Γ are the form r from §2.5.1 with domain Ω and Γ respectively. By the bounds on r_Ω, r_Γ and λ, we have

$$\begin{aligned}
|r(t;\mathbf{v_1})| &\le C_1(\|\omega_1\|^2_{H^1(\Omega(t))} + \|\gamma_1\|^2_{H^1(\Gamma(t))} + \|\beta\gamma_1-\alpha\omega_1\|^2_{L^2(\Gamma(t))})\\
&\le C_2(\|\omega_1\|^2_{H^1(\Omega(t))} + \|\gamma_1\|^2_{H^1(\Gamma(t))} + \|\gamma_1\|^2_{L^2(\Gamma(t))} + \|\omega_1\|^2_{L^2(\Gamma(t))})\\
&\le C_2((1+C_T^2)\,\|\omega_1\|^2_{H^1(\Omega(t))} + 2\,\|\gamma_1\|^2_{H^1(\Gamma(t))})\\
&\le C_3\,\|\mathbf{v_1}\|^2_{V(t)}\,,
\end{aligned}$$

i.e. $r(t;\cdot)$ is bounded in $V(t)$. With all the assumptions satisfied, we find from Theorem 1.4.8 that there is a unique solution $(u,v)\in W(V,H)$ to the problem $(\mathbf{P_{bs}})$.

2.5.4 The dynamic boundary problem for an elliptic equation (2.11)

We are going to formulate the problem (2.11) as a parabolic equation on $\Gamma(t)$. Note that $v(t)$ has a normal derivative (we expect $v(t)\in H^1(\Omega(t))$ and since $\Delta v(t)=0$) and so we can define using (2.1) the **Dirichlet-to-Neumann map** $\mathbb{A}(t)\colon H^{\frac{1}{2}}(\Gamma(t))\to H^{-\frac{1}{2}}(\Gamma(t))$ (which is also bounded) by

$$\mathbb{A}(t)u(t) = \frac{\partial v(t)}{\partial\nu(t)}.$$

This map is also commonly known as the Poincaré–Steklov operator in the theory of boundary integral equations [109, §3.7]. Now, define $\mathbb{D}(t)\colon H^{\frac{1}{2}}(\Gamma(t))\to H^1(\Omega(t))$ by $\mathbb{D}(t)\tilde{u}=\tilde{v}$ where $\tilde{v}$ is the unique weak solution of

$$\begin{aligned}
\Delta\tilde{v} &= 0 \quad \text{on } \Omega(t)\\
\tilde{v} &= \tilde{u} \quad \text{on } \Gamma(t)
\end{aligned} \tag{2.19}$$

given $\tilde{u}\in H^{\frac{1}{2}}(\Gamma(t))$. These maps give us a clue as to the spaces where we should look for solutions. Formally, we may think of a solution of the PDE (2.11) as a pair

$(v, u) \in L^2_{H^1} \times W(H^{\frac{1}{2}}, H^{-\frac{1}{2}})$ such that given $f \in L^2_{H^{-\frac{1}{2}}}$,

$$\begin{aligned} v &= \mathbb{D}u && \text{in } L^2_{H^1} \\ \dot{u} + \mathbb{A}u + u &= f && \text{in } L^2_{H^{-\frac{1}{2}}} \\ u(0) &= v_0 && \text{in } L^2(\Gamma_0) \end{aligned} \tag{2.20}$$

holds. Note that $(\mathbb{D}u)(t) = \mathbb{D}(t)u(t)$ for a.e. t. Of course, we have not defined these spaces yet so this is just formal as mentioned.

Function spaces

We use the notation and the established results of §2.4.1. We assume some stronger regularity on the map Φ^0_t here, namely

$$\Phi^0_t \colon \Gamma_0 \to \Gamma(t) \text{ is a } C^3\text{-diffeomorphism} \qquad \text{and} \qquad \Phi^0_{(\cdot)} \in C^3([0, T] \times \Gamma_0).$$

In this case, we use the pivot space $H(t) = L^2(\Gamma(t))$ but now require $V(t) = H^{\frac{1}{2}}(\Gamma(t))$. Below, we shall mainly make use of $\phi_{\Gamma,t}$ and to save space we shall write it simply as ϕ_t. We only revert to the full notation when ambiguity forces us to.

We already know that $\phi_{-t} \colon L^2(\Gamma(t)) \to L^2(\Gamma_0)$ is a well-defined linear homeomorphism. Now we show that the map $\phi_{-t} \colon H^{\frac{1}{2}}(\Gamma(t)) \to H^{\frac{1}{2}}(\Gamma_0)$ is also a linear homeomorphism. Letting $u \in H^{\frac{1}{2}}(\Gamma(t))$, it suffices to estimate only the seminorm $|\phi_{-t}u|_{H^{\frac{1}{2}}(\Gamma_0)}$:

$$\int_{\Gamma_0}\int_{\Gamma_0} \frac{|\phi_{-t}u(x) - \phi_{-t}u(y)|^2}{|x-y|^n} = \int_{\Gamma(t)}\int_{\Gamma(t)} \frac{|u(x_t) - u(y_t)|^2}{|\Phi^t_0(x_t) - \Phi^t_0(y_t)|^n} J^t_0(x_t) J^t_0(y_t) \tag{2.21}$$

where we made the substitutions $x_t = \Phi^0_t(x) \in \Gamma(t)$ and $y_t = \Phi^0_t(y) \in \Gamma(t)$. Since Φ^0_t is a C^1-diffeomorphism between compact spaces, it is bi-Lipschitz with Lipschitz constant C_L independent of t (because the spatial derivatives of Φ^0_t are uniformly bounded). This implies $|x_t - y_t| \le C_L|\Phi^t_0(x_t) - \Phi^t_0(y_t)|$ so that (2.21) becomes

$$|\phi_{-t}u|^2_{H^{\frac{1}{2}}(\Gamma_0)} \le C^n_L C^2_J \int_{\Gamma(t)}\int_{\Gamma(t)} \frac{|u(x_t) - u(y_t)|^2}{|x_t - y_t|^n} = C^n_L C^2_J |u|^2_{H^{\frac{1}{2}}(\Gamma(t))},$$

where we used the uniform bound on J^t_0. So we have the uniform bound

$$\|\phi_{-t}u\|_{H^{\frac{1}{2}}(\Gamma_0)} \le C \|u\|_{H^{\frac{1}{2}}(\Gamma(t))}.$$

A similar bound holds for the operator ϕ_t by the same arguments as above since

$\Phi_0^t = (\Phi_t^0)^{-1}$ also satisfies the same properties as above. It follows by the smoothness on $\Phi_{(\cdot)}^0$ that $J_{(\cdot)}^0 \in C^2([0,T] \times \Gamma_0)$. This implies that $J_t^0 \colon \Gamma_0 \to \mathbb{R}$ is (globally) Lipschitz (see the paragraph after the proof of Proposition 2.4 in [73]). The map

$$t \mapsto |\phi_t u|^2_{H^{\frac{1}{2}}(\Gamma(t))} = \int_{\Gamma_0} \int_{\Gamma_0} \frac{|u(x_0) - u(y_0)|^2}{|\Phi_t^0(x_0) - \Phi_t^0(y_0)|^n} J_t^0(x_0) J_t^0(y_0)$$

is continuous. To see this, define the integrand

$$g(x_0, y_0, t) = \frac{|u(x_0) - u(y_0)|^2}{|\Phi_t^0(x_0) - \Phi_t^0(y_0)|^n} J_t^0(x_0) J_t^0(y_0).$$

Now, $t \mapsto g(x_0, y_0, t)$ is continuous for almost all (x_0, y_0) (it only fails when the denominator is zero, where $x_0 = y_0$, and the set of such points has zero measure), and we have the domination $g(x_0, y_0, t) \leq h(x_0, y_0)$ for all t and almost all (x_0, y_0) by an integrable function h; this follows due to the smoothness assumptions on $\Phi_{(\cdot)}^0$ and $J_{(\cdot)}^0$. Therefore, $t \mapsto \int_{\Gamma_0} \int_{\Gamma_0} g(x_0, y_0, t)$ is continuous. This enables us to conclude that $(H, \phi_{(\cdot)})$ and $(V, \phi_{(\cdot)}|_V)$ are compatible.

There is some effort needed in order to show the evolving space equivalence. We start with the following two results which are used continually.

Lemma 2.5.2. For $y \in \Gamma_0$, we have

$$\int_{\Gamma_0} \frac{1}{|x - y|^{n-2}} \,\mathrm{d}\sigma(x) < C$$

where C does not depend on y.

This lemma can be proved by first setting $y = 0$ (without loss of generality) and then splitting the domain of integration into two sets, one of which is a ball centred at the origin. The integral over the ball can be tackled with the assumption of the domain being Lipschitz and switching to polar coordinates, while the integral over the complement of the ball is obviously finite.

Lemma 2.5.3. If $\rho \in C^1(\Gamma_0)$ and $u \in H^{\frac{1}{2}}(\Gamma_0)$ then $\rho u \in H^{\frac{1}{2}}(\Gamma_0)$ and

$$\|\rho u\|_{H^{\frac{1}{2}}(\Gamma_0)} \leq C \|\rho\|_{C^1(\Gamma_0)} \|u\|_{H^{\frac{1}{2}}(\Gamma_0)} \tag{2.22}$$

where C does not depend on ρ or u.

Proof. Note that ρ and $\nabla\rho$ are bounded from above and ρ is Lipschitz. We begin with

$$\|\rho u\|^2_{H^{\frac{1}{2}}(\Gamma_0)} \leq \|\rho\|^2_{C^0(\Gamma_0)} \|u\|^2_{L^2(\Gamma_0)} + \int_{\Gamma_0}\int_{\Gamma_0} \frac{|\rho(x)u(x)-\rho(y)u(y)|^2}{|x-y|^n}\,\mathrm{d}x\mathrm{d}y.$$

The last term is

$$\begin{aligned}
&\int_{\Gamma_0}\int_{\Gamma_0} \frac{|\rho(x)u(x)-\rho(y)u(y)|^2}{|x-y|^n} \\
&\quad\leq 2\int_{\Gamma_0}\int_{\Gamma_0} \frac{|\rho(x)|^2|u(x)-u(y)|^2}{|x-y|^n} + 2\int_{\Gamma_0}\int_{\Gamma_0} \frac{|u(y)|^2|\rho(x)-\rho(y)|^2}{|x-y|^n} \\
&\quad\leq 2\|\rho\|^2_{C^0(\Gamma_0)} |u|^2_{H^{\frac{1}{2}}(\Gamma_0)} + 2\|\nabla\rho\|^2_{C^0(\Gamma_0)} \int_{\Gamma_0}\int_{\Gamma_0} \frac{|u(y)|^2}{|x-y|^{n-2}}.
\end{aligned}$$

Using the previous lemma, the integral in the second term is

$$\int_{\Gamma_0}\int_{\Gamma_0} \frac{|u(y)|^2}{|x-y|^{n-2}} = \int_{\Gamma_0} |u(y)|^2 \int_{\Gamma_0} |x-y|^{2-n} \leq C_1 \|u\|^2_{L^2(\Gamma_0)}.$$

Putting it all together, we achieve (2.22). □

In the following lemmas, let $J \in C^2([0,T]\times\Gamma_0)$.

Lemma 2.5.4. If $\psi \in \mathcal{D}((0,T); H^{\frac{1}{2}}(\Gamma_0))$, then $\psi J \in \mathcal{W}(V_0, V_0^*)$ and $(\psi J)' = \psi' J + \psi J'$.

Proof. Let us note that

$$\psi \in C^0([0,T]; H^{\frac{1}{2}}(\Gamma_0)) \quad \text{and} \quad J \in C^0([0,T]; H^{\frac{1}{2}}(\Gamma_0)) \cap C^1([0,T]; C^1(\Gamma_0)).$$

The first part of the second inclusion holds because $J \in C^0([0,T]; H^1(\Gamma_0))$ and because $H^1(\Gamma_0) \subset H^{\frac{1}{2}}(\Gamma_0)$ is continuous [109, Theorem 2.5.1 and Theorem 2.5.5]. The uniform continuity of J over the compact set $[0,T]\times\Gamma_0$ gives the second part.

Now, note that $\psi(t)J(t) \in H^{\frac{1}{2}}(\Gamma_0)$ for all t by Lemma 2.5.3. To see that $\psi J \in C^0([0,T]; H^{\frac{1}{2}}(\Gamma_0))$, fix an arbitrary $t \in [0,T]$, let $t_n \to t$ and consider

$$\begin{aligned}
\|\psi(t)J(t) - \psi(t_n)J(t_n)\|_{H^{\frac{1}{2}}(\Gamma_0)} &\leq \|\psi(t)(J(t)-J(t_n))\|_{H^{\frac{1}{2}}(\Gamma_0)} \\
&\quad + \|J(t_n)(\psi(t)-\psi(t_n))\|_{H^{\frac{1}{2}}(\Gamma_0)} \\
&\leq C\|J(t)-J(t_n)\|_{C^1(\Gamma_0)} \|\psi(t)\|_{H^{\frac{1}{2}}(\Gamma_0)} \\
&\quad + C\|J(t_n)\|_{C^1(\Gamma_0)} \|\psi(t)-\psi(t_n)\|_{H^{\frac{1}{2}}(\Gamma_0)}.
\end{aligned}$$

The first of these terms tends to zero as $t_n \to t$ because $J \in C^0([0,T]; C^1(\Gamma_0))$ and the second because $\psi \in C^0([0,T]; H^{\frac{1}{2}}(\Gamma_0))$ in addition to the aforementioned smoothness of J.

Now we show that in fact ψJ is (classically) differentiable and that $(\psi J)' = \psi' J + \psi J'$. Observe that $\psi'(t)J(t) + \psi(t)J'(t) \in H^{\frac{1}{2}}(\Gamma_0)$ by Lemma 2.5.3. Define the difference quotient $D^h J(t) = (J(t+h) - J(t))/h$ and $D^h\psi(t)$ similarly and note that

$$\begin{aligned}
&\left\| \frac{\psi(t+h)J(t+h) - \psi(t)J(t)}{h} - \psi'(t)J(t) - \psi(t)J'(t) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} \\
&\leq \left\| \psi(t+h)D^h J(t) - \psi(t)J'(t) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} + \left\| D^h\psi(t)J(t) - \psi'(t)J(t) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} \\
&\leq C \left\| D^h J(t) - J'(t) \right\|_{C^1(\Gamma_0)} \|\psi(t+h)\|_{H^{\frac{1}{2}}(\Gamma_0)} \\
&\quad + C \left\| J'(t) \right\|_{C^1(\Gamma_0)} \|\psi(t+h) - \psi(t)\|_{H^{\frac{1}{2}}(\Gamma_0)} \\
&\quad + C \|J(t)\|_{C^1(\Gamma_0)} \left\| D^h\psi(t) - \psi'(t) \right\|_{H^{\frac{1}{2}}(\Gamma_0)}.
\end{aligned}$$

In the above, we used

$$\begin{aligned}
\left\| \psi(t+h)D^h J(t) - \psi(t)J'(t) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} &\leq \left\| \psi(t+h) \left(D^h J(t) - J'(t) \right) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} \\
&\quad + \left\| (\psi(t+h) - \psi(t))J'(t) \right\|_{H^{\frac{1}{2}}(\Gamma_0)}.
\end{aligned}$$

It follows that $\left\| D^h J(t) - J'(t) \right\|_{C^1(\Gamma_0)} \to 0$ because $J \in C^1([0,T]; C^1(\Gamma_0))$. Thus, we find

$$\lim_{h\to 0} \left\| \frac{\psi(t+h)J(t+h) - \psi(t)J(t)}{h} - \psi'(t)J(t) - \psi(t)J'(t) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} = 0.$$

This proves the product rule for $(\psi J)'$. Now we finish by proving that $(\psi J)' \in C^0([0,T]; H^{\frac{1}{2}}(\Gamma_0))$. Fix again $t \in [0,T]$ and let $t_n \to t$. Observe that

$$\begin{aligned}
&\left\| \psi'(t_n)J(t_n) + \psi(t_n)J'(t_n) - \psi'(t)J(t) - \psi(t)J'(t) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} \\
&\leq \left\| \psi'(t_n)(J(t_n) - J(t)) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} + \left\| J(t)(\psi'(t_n) - \psi'(t)) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} \\
&\quad + \left\| \psi(t_n)(J'(t_n) - J'(t)) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} + \left\| J'(t)(\psi(t_n) - \psi(t)) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} \\
&\leq C \left\| \psi'(t_n) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} \|J(t_n) - J(t)\|_{C^1(\Gamma_0)} + C \|J(t)\|_{C^1(\Gamma_0)} \left\| \psi'(t_n) - \psi'(t) \right\|_{H^{\frac{1}{2}}(\Gamma_0)} \\
&\quad + C \|\psi(t_n)\|_{H^{\frac{1}{2}}(\Gamma_0)} \left\| J'(t_n) - J'(t) \right\|_{C^1(\Gamma_0)} + C \left\| J'(t) \right\|_{C^1(\Gamma_0)} \|\psi(t_n) - \psi(t)\|_{H^{\frac{1}{2}}(\Gamma_0)}
\end{aligned}$$

and this tends to zero because $J \in C^1([0,T]; C^1(\Gamma_0))$ and $\psi \in C^1([0,T]; H^{\frac{1}{2}}(\Gamma_0))$. All in all, we have shown that $\psi J \in C^1([0,T]; H^{\frac{1}{2}}(\Gamma_0)) \subset \mathcal{W}(V_0, V_0^*)$. $\square$

Lemma 2.5.5. For every $u \in \mathcal{W}(V_0, V_0^*)$, $Ju \in \mathcal{W}(V_0, V_0^*)$.

Proof. Let $\psi \in \mathcal{D}((0,T); H^{\frac{1}{2}}(\Gamma_0))$ and for $u \in \mathcal{W}(V_0, V_0^*)$, consider

$$\begin{aligned}\int_0^T \langle u'(t), J(t)\psi(t)\rangle_{H^{-\frac{1}{2}}(\Gamma_0), H^{\frac{1}{2}}(\Gamma_0)} &= -\int_0^T (J'(t)\psi(t) + J(t)\psi'(t), u(t))_{L^2(\Gamma_0)} \\ &\quad \text{(by integration by parts and the last lemma)} \\ &= -\int_0^T (\psi(t), J'(t)u(t))_{L^2(\Gamma_0)} \\ &\quad - \int_0^T (\psi'(t), J(t)u(t))_{L^2(\Gamma_0)}.\end{aligned}$$

A rearrangement yields

$$\int_0^T (J(t)u(t), \psi'(t))_{L^2(\Gamma_0)} = -\int_0^T \langle J'(t)u(t) + J(t)u'(t), \psi(t)\rangle_{H^{-\frac{1}{2}}(\Gamma_0), H^{\frac{1}{2}}(\Gamma_0)}.$$

This shows that Ju has a weak derivative, and $(Ju)' \in L^2(0,T; H^{-\frac{1}{2}}(\Gamma_0))$ since we have $J'u \in L^2(0,T; H^{\frac{1}{2}}(\Gamma_0))$ and $Ju' \in L^2(0,T; H^{-\frac{1}{2}}(\Gamma_0))$. $\square$

Theorem 2.5.6. The evolving space equivalence between $\mathcal{W}(V_0, V_0^*)$ and $W(V, V^*)$ holds.

Proof. The last result shows that if $u \in \mathcal{W}(V_0, V_0^*)$ then $J_t^0 u \in \mathcal{W}(V_0, V_0^*)$. Because $1/J_t^0 \in C^2([0,T] \times \Gamma_0)$, the converse also holds. Since

$$(J_t^0 u(t))' = J_t^0 u'(t) + \hat{\Lambda}(t)u(t),$$

we have (in the notation of Theorem 1.2.46) $\hat{S}(t) = T_t = J_t^0$ and $\hat{D}(t) \equiv 0$, and it follows that $\hat{S}(\cdot)u'(\cdot) \in L^2(0,T; H^{-\frac{1}{2}}(\Gamma_0))$. Thus Theorem 1.2.46 can be applied. $\square$

Weak formulation and well-posedness

Now that we have defined some notation and function spaces, the equation (2.20) has a precise meaning and we can define a notion of solution.

Definition 2.5.7. With $H^1 = \{H^1(\Omega(t))\}_{t\in[0,T]}$, given $f \in L^2_{V^*}$, a solution of (2.11) is a pair $(v, u) \in L^2_{H^1} \times W(V, V^*)$ such that

$$\begin{aligned} v &= \mathbb{D}u && \text{in } L^2_{H^1} \\ \dot{u} + \mathbb{A}u + u &= f && \text{in } L^2_{V^*} \\ u(0) &= v_0 && \text{in } H_0. \end{aligned} \tag{2.23}$$

Note that the first condition implies $\Delta_t v(t) = 0$ and $v(t)|_{\Gamma(t)} = u(t)$ for almost every t. We need the following auxiliary result.

Lemma 2.5.8. The map $\mathbb{D}(t)\colon H^{\frac{1}{2}}(\Gamma(t)) \to H^1(\Omega(t))$ is uniformly bounded:

$$\|\mathbb{D}(t)\tilde{u}\|_{H^1(\Omega(t))} \leq C\,\|\tilde{u}\|_{H^{\frac{1}{2}}(\Gamma(t))} \qquad \forall \tilde{u} \in H^{\frac{1}{2}}(\Gamma(t)) \tag{2.24}$$

where the constant C does not depend on $t \in [0, T]$.

To prove this lemma, we need the following results which show that certain standard results are in a sense uniform in $t \in [0, T]$. The method of proof of the next lemma is identical to that of Lemma 2.4.2.

Lemma 2.5.9. Let $\tau_t\colon H^1(\Omega(t)) \to H^{\frac{1}{2}}(\Gamma(t))$ denote the trace map. For all $v \in H^1(\Omega_0)$, the equality $\tau_t(\phi_{\Omega,t}v) = \phi_{\Gamma,t}(\tau_0 v)$ holds in $H^{\frac{1}{2}}(\Gamma(t))$.

Lemma 2.5.10. For each $t \in [0, T]$, we have

$$\|v\|_{H^1(\Omega(t))} \leq C_1\,\|\nabla v\|_{L^2(\Omega(t))} \qquad \forall v \in H^1_0(\Omega(t)) \tag{2.25}$$

$$\|\nabla v\|^2_{L^2(\Omega(t))} + \|v\|^2_{L^2(\Gamma(t))} \geq C_2\,\|v\|^2_{H^1(\Omega(t))} \qquad \forall v \in H^1(\Omega(t)) \tag{2.26}$$

$$\inf_{\substack{v\in H^1(\Omega(t))\\ \tau_t v = u}} \|v\|_{H^1(\Omega(t))} \leq C_3\,\|u\|_{H^{\frac{1}{2}}(\Gamma(t))} \qquad \forall u \in H^{\frac{1}{2}}(\Gamma(t)) \tag{2.27}$$

$$\|\tau_t v\|_{H^{\frac{1}{2}}(\Gamma(t))} \leq C_4\,\|v\|_{H^1(\Omega(t))} \qquad \forall v \in H^1(\Omega(t)) \tag{2.28}$$

where C_1, C_2, C_3, and C_4 do not depend on t.

The strategy to prove this lemma is to start with each respective inequality at $t = 0$, in which case: (2.25) is the Poincaré inequality on Ω_0, (2.26) follows by a compactness argument, (2.27) is an equivalence of norms and (2.28) is the trace inequality on Ω_0. Then for (2.25), use the chain rule $\nabla(\phi_{-t}v) = \nabla(v \circ \Phi^0_t) = \phi_{-t}(\nabla v)\mathbf{D}\Phi^0_t$ and the uniform boundedness of $\mathbf{D}\Phi^0_t$. The inequality (2.26) is obtained with the identity $\nabla v = \nabla(\phi_{-t}\phi_t v) = \phi_{-t}(\nabla\phi_t v)\mathbf{D}\Phi^0_t$ and Lemma 2.5.9. The lemma is also the key ingredient to show (2.27) and (2.28) (see the discussion in §2.4.2 for how to prove the latter).

Proof of Lemma 2.5.8. We prove the well-posedness of (2.19) in addition to the uniform bound (2.24) for the convenience of the reader. First, we use the trace map $\tau_t\colon H^1(\Omega(t)) \to H^{\frac{1}{2}}(\Gamma(t))$ to see that there is a function $\tilde{v}_{\tilde{u}} \in H^1(\Omega(t))$ such that $\tau_t \tilde{v}_{\tilde{u}} = \tilde{u}$. With $\tilde{v} = \mathbb{D}(t)\tilde{u}$, set $d := \tilde{v} - \tilde{v}_{\tilde{u}}$. Then d solves

$$\begin{aligned} \Delta d &= -\Delta \tilde{v}_{\tilde{u}} && \text{on } \Omega(t) \\ d &= 0 && \text{on } \Gamma(t). \end{aligned} \tag{2.29}$$

Define $b_t(\cdot,\cdot)\colon H^1(\Omega(t)) \times H^1(\Omega(t)) \to \mathbb{R}$ and $l_t(\cdot)\colon H^1(\Omega(t)) \to \mathbb{R}$ by

$$b_t(d,\varphi) = \int_{\Omega(t)} \nabla d \nabla \varphi \qquad \text{and} \qquad l_t(\varphi) = \int_{\Omega(t)} \nabla \tilde{w}_{\tilde{u}} \nabla \varphi.$$

Clearly l_t and b_t are bounded and the Poincaré inequality (2.25) implies that b_t is coercive with the coercivity constant C_P^{-1} independent of t. By Lax–Milgram, there is a unique solution $d \in H^1_0(\Omega(t))$ to (2.29) satisfying

$$\|d\|_{H^1(\Omega(t))} \le C_P \|\tilde{v}_{\tilde{u}}\|_{H^1(\Omega(t))}.$$

Because this inequality holds for all lifts $\tilde{v}_{\tilde{u}}$ of $\tilde{u}$ we must have

$$\begin{aligned} \|d\|_{H^1(\Omega(t))} &\le C_P \inf_{\substack{w \in H^1(\Omega(t)),\\ \tau_t w = \tilde{u}}} \|w\|_{H^1(\Omega(t))} \\ &\le C_1 \|\tilde{u}\|_{H^{\frac{1}{2}}(\Gamma(t))} \end{aligned}$$

where the second inequality is thanks to (2.27). Since $\tilde{v} = d + \tilde{v}_{\tilde{u}}$, we see that (2.19) has a unique solution $\tilde{v} \in H^1(\Omega(t))$ with

$$\|\tilde{v}\|_{H^1(\Omega(t))} \le C_2 \|\tilde{u}\|_{H^{\frac{1}{2}}(\Gamma(t))}$$

due to the arbitrariness of the lift $\tilde{v}_{\tilde{u}}$. □

Now we can conclude the well-posedness of (2.23) by checking the assumptions on $\mathbb{A}$. With $w \in L^2_V$ and using (2.1),

$$\langle \mathbb{A}(t)u(t), w(t)\rangle_{H^{-\frac{1}{2}}(\Gamma(t)),H^{\frac{1}{2}}(\Gamma(t))} = \int_{\Omega(t)} \nabla(\mathbb{D}(t)u(t)) \nabla(\mathbb{E}(t)w(t)).$$

So the bilinear form $a(t;\cdot,\cdot)\colon H^{\frac{1}{2}}(\Gamma(t)) \times H^{\frac{1}{2}}(\Gamma(t)) \to \mathbb{R}$ is

$$a(t;u,w) := \int_{\Omega(t)} \nabla(\mathbb{D}(t)u)\nabla(\mathbb{E}(t)w) + \int_{\Gamma(t)} uw.$$

We take $\mathbb{E} = \mathbb{D}$, and we obtain by the uniform bound (2.24) the boundedness of $a(t;\cdot,\cdot)$:

$$\begin{aligned}
|a(t;u,w)| &\leq \|\mathbb{D}(t)u\|_{H^1(\Omega(t))} \|\mathbb{D}(t)w\|_{H^1(\Omega(t))} + \|u\|_{L^2(\Gamma(t))} \|w\|_{L^2(\Gamma(t))}\\
&\leq C_D^2 \|u\|_{H^{\frac{1}{2}}(\Gamma(t))} \|w\|_{H^{\frac{1}{2}}(\Gamma(t))} + \|u\|_{L^2(\Gamma(t))} \|w\|_{L^2(\Gamma(t))}\\
&\leq (C_D^2 + 1) \|u\|_{H^{\frac{1}{2}}(\Gamma(t))} \|w\|_{H^{\frac{1}{2}}(\Gamma(t))}.
\end{aligned}$$

For coercivity,

$$\begin{aligned}
a(t;w,w) &= \|\nabla(\mathbb{D}(t)w)\|_{L^2(\Omega(t))}^2 + \|w\|_{L^2(\Gamma(t))}^2 && \text{(again with } \mathbb{E} = \mathbb{D}\text{)}\\
&\geq C_1 \|\mathbb{D}(t)w\|_{H^1(\Omega(t))}^2 && \text{(using (2.26))}\\
&\geq C_2 \|w\|_{H^{\frac{1}{2}}(\Gamma(t))}^2
\end{aligned}$$

by the trace inequality (2.28). Thus, there is a unique solution $u \in W(H^{\frac{1}{2}}, H^{-\frac{1}{2}})$ to (2.23), and with $v(t) := \mathbb{D}(t)u(t)$ and the uniform bound (2.24), we find (v,u) to be a solution of (2.11).

Remark 2.5.11. Lions in [84, §1.11.1] treats this problem on a stationary domain with a lower order nonlinear term $|u|^p u$, and proves existence with a Galerkin method in the space $L^2(0,T;H^{\frac{1}{2}}(\Gamma)) \cap L^\infty(0,T;L^2(\Gamma)) \cap L^{p+2}(\Gamma \times (0,T))$ by using the Dirichlet-to-Neumann characterisation. The stationary case there (at least with $p = 0$) is much more straightforward since the functional analytic results we developed in §2.5.4 would not be needed.

Chapter 3

A Stefan problem on an evolving surface

3.1 Introduction

The Stefan problem is the prototypical time-dependent free boundary problem and is the canonical mathematical model describing phase change of a substance. It arises in various forms in many models in the physical and biological sciences [54, 65, 90, 104]. It is named after Josef Stefan who first formulated the general class of problems in 1889–1891, and the associated theory of weak solutions was studied in [97, 77] by Oleĭnik and Kamenomostskaja seventy years later. The model to have in mind is water and ice on a domain; where the temperature of the substance is less than zero, it is said to be in the solid phase, and when it is greater than zero, it is in the liquid phase. The subset of the domain that separates the two phases is called the *interface*. There are diffusion processes in the two phases and a boundary condition on the interface (known as the *Stefan condition*), and the evolution in time of the interface is unknown (as it is defined by the temperature, which is the solution of the equation) and thus is a *free boundary*. Common research questions involve regularity of the solution and the free boundary [64, 111] — is the free boundary a hypersurface, and if so, how smooth is it? These are interesting issues, but their delicacy leaves them beyond the scope of this work. In this chapter we present a theory of weak solutions associated with the so-called *enthalpy* approach [54] to the Stefan problem on an evolving curved hypersurface.

Our interest is in the existence, uniqueness and continuous dependence of

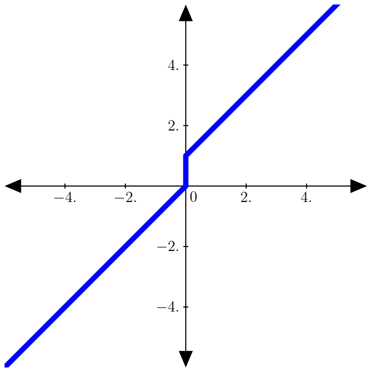

Figure 3.1.1: The graph $\mathcal{E}$

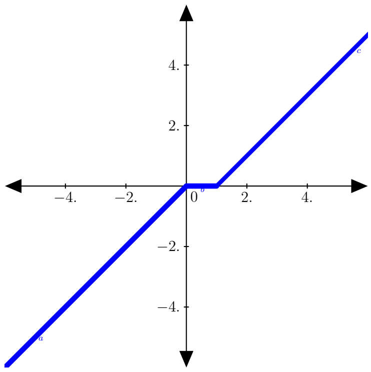

Figure 3.1.2: The map $\mathcal{E}^{-1}$

weak solutions to

$$\begin{aligned} \partial^\bullet e(t) - \Delta_{\Omega(t)} u(t) + e(t)\nabla_{\Omega(t)} \cdot \mathbf{w}(t) &= f(t) \quad \text{in } \Omega(t) \\ e(0) &= e_0 \quad \text{on } \Omega(0) \\ e &\in \mathcal{E}(u) \end{aligned} \tag{3.1}$$

posed on a moving compact hypersurface $\Omega(t) \subset \mathbb{R}^{n+1}$ evolving with (given) velocity field $\mathbf{w}$, where the energy $\mathcal{E} \colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is defined by

$$\mathcal{E}(r) = \begin{cases} r & \text{for } r < 0 \\ [0,1] & \text{for } r = 0 \\ r+1 & \text{for } r > 0. \end{cases} \tag{3.2}$$

The jump at the origin of the energy (or enthalpy) $\mathcal{E}$ corresponds to the phase transition at the zero temperature. Note that $\mathcal{E}$ is a maximal monotone graph in the sense of Brézis [26]. Problem (3.1) is indeed a Stefan problem, as we shall see below shortly.

The novelty of this work is that the Stefan problem itself is formulated on a moving hypersurface and our chosen method to treat this problem, which we believe is naturally suited to equations on moving domains, requires the use of some new functional analysis results that we shall introduce, building upon the concepts presented in Chapters 1 and 2. There is, as alluded to above, a rich literature associated to Stefan-type problems [15, 64, 77, 97, 106, 105]. We will show that arguments similar to those used in the standard setting (for example, mollifying the nonlinearity, linearisation of the resulting PDE, applying a fixed point theorem, and then passing to the limit) are also amenable to our problem on a moving hypersurface, thanks in part to the function spaces we decide to use. An important contribution of this work is that it shows how the time-evolving spaces can be handled with relative ease thanks to the theory of Chapters 1 and 2 and reveals the type of arguments one needs to make in this setting. Let us remark that the techniques and functional analysis we develop here can be directly applied to study many other nonlinear PDE problems posed on moving domains.

Let us work out a possible pointwise formulation of (3.1) and relate it to what we have described in the introduction. Start by supposing $\Omega(t) = \Omega_l(t) \cup \Omega_s(t) \cup \Gamma(t)$ where $\Omega_l(t)$ and $\Omega_s(t)$ divide $\Omega(t)$ into a liquid and a solid phase (respectively) with an *a priori* unknown interface $\Gamma(t)$. The quantity of interest is the temperature $u(t)\colon \Omega(t) \to \mathbb{R}$, which we suppose satisfies

$$\begin{cases} u(t) > 0 & \text{in } \Omega_l(t) \\ u(t) = 0 & \text{in } \Gamma(t) \\ u(t) < 0 & \text{in } \Omega_s(t), \end{cases}$$

and thus $u = 0$ is the critical temperature where the change of phase occurs. Define

$$Q_l = \bigcup_{t\in(0,T)} \Omega_l(t) \times \{t\}, \qquad S = \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\},$$

and Q_s similarly. Given f and u_0, we formally elucidate in Remark 3.2.5 the relationship between (3.1) and the following model describing the temperature u:

$$\begin{aligned} \partial^\bullet u - \Delta_\Omega u + (u+1)\nabla_\Omega \cdot \mathbf{w} &= f && \text{in } Q_l \\ \partial^\bullet u - \Delta_\Omega u + u\nabla_\Omega \cdot \mathbf{w} &= f && \text{in } Q_s \\ -(\nabla_\Omega u_l - \nabla_\Omega u_s) \cdot \mu &= V && \text{on } S \\ u &= 0 && \text{on } S \\ u(0) &= u_0 && \text{on } \Omega(0), \end{aligned} \tag{3.3}$$

where u_s denotes the trace of the restriction $u|_{\Omega_s}$ to the interface Γ (likewise with u_l), $V(t)$ is the conormal velocity of $\Gamma(t)$ and $\mu(t)$ is the unit conormal vector pointing into $\Omega_l(t)$ (this vector is tangential to $\Omega(t)$ and normal to $\partial\Omega_l(t)$).

We now introduce some notions of a weak solution, similar to [77].

Definition 3.1.1 (Weak solution)**.** Given $f \in L^1_{L^1}$ and $e_0 \in L^1(\Omega_0)$, a *weak solution* of (3.1) is a pair $(u,e) \in L^1_{L^1} \times L^1_{L^1}$ such that $e \in \mathcal{E}(u)$ and there holds

$$-\int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) - \int_0^T \int_{\Omega(t)} u(t)\Delta_\Omega \eta(t) = \int_0^T \int_{\Omega(t)} f(t)\eta(t) + \int_{\Omega_0} e_0\eta(0)$$

for all $\eta \in W(L^\infty \cap H^2, L^\infty)$ with $\Delta_\Omega \eta \in L^\infty_{L^\infty}$ and $\eta(T) = 0$.

In order to carry out the well-posedness proof, we will use the following stronger notion of a weak solution too.

Definition 3.1.2 (Bounded weak solution)**.** Given $f \in L^\infty_{L^\infty}$ and $e_0 \in L^\infty(\Omega_0)$, a *bounded weak solution* of (3.1) is a pair $(u,e) \in L^2_{H^1} \times L^\infty_{L^\infty}$ such that (u,e) is a weak solution of (3.1) satisfying

$$-\int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) + \int_0^T \int_{\Omega(t)} \nabla_\Omega u(t) \nabla_\Omega \eta(t) = \int_0^T \int_{\Omega(t)} f(t)\eta(t) + \int_{\Omega_0} e_0\eta(0) \quad (3.4)$$

for all $\eta \in W(H^1, L^2)$ with $\eta(T) = 0$.

We first prove well-posedness of bounded weak solutions for bounded data.

Theorem 3.1.3 (Existence of bounded weak solutions)**.** If $f \in L^\infty_{L^\infty}$, $e_0 \in L^\infty(\Omega_0)$ and $|\Omega| := \sup_{s\in[0,T]} |\Omega(s)| < \infty$, then there exists a bounded weak solution to (3.1).

Theorem 3.1.4 (Uniqueness and continuous dependence of bounded weak solutions)**.** If for $i = 1, 2$, (u^i, e^i) are two bounded weak solutions of (3.1) with data $(f^i, e^i_0) \in L^\infty_{L^\infty} \times L^\infty(\Omega_0)$, then

$$\|e^1(t) - e^2(t)\|_{L^1(\Omega(t))} \le \int_0^t \|f^1(\tau) - f^2(\tau)\|_{L^1(\Omega(\tau))} + \|e^1_0 - e^2_0\|_{L^1(\Omega_0)}$$

for almost all t.

The continuous dependence of the previous theorem allows us to extend the well-posedness to data belonging to the integrable class through an approximation argument. This is given by the next theorem, which is the main result of this chapter.

Theorem 3.1.5 (Well-posedness of weak solutions)**.** If $f \in L^1_{L^1}$, $e_0 \in L^1(\Omega_0)$ and $|\Omega| := \sup_{s\in[0,T]} |\Omega(s)| < \infty$, then there exists a unique weak solution to (3.1). Furthermore, if for $i = 1, 2$, $(u^i, e^i) \in L^1_{L^1} \times L^1_{L^1}$ are two weak solutions of (3.1) with data $(f^i, e^i_0) \in L^1_{L^1} \times L^1(\Omega_0)$, then

$$\|e^1 - e^2\|_{L^1_{L^1}} \le C_T \left(\|f^1 - f^2\|_{L^1_{L^1}} + \|e^1_0 - e^2_0\|_{L^1(\Omega_0)} \right).$$

3.2 Preliminaries

3.2.1 Function spaces on evolving surfaces

We now make precise the assumptions on the evolving surface $\Omega(t)$ our Stefan problem is posed on and we discuss function spaces. For each $t \in [0,T]$, let $\Omega(t) \subset \mathbb{R}^{n+1}$ be an orientable compact (i.e., no boundary) n-dimensional hypersurface of class C^3, and assume the existence of a flow $\Phi\colon [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that for all $t \in [0,T]$, with $\Omega_0 := \Omega(0)$, the map $\Phi_t^0(\cdot) := \Phi(t,\cdot)\colon \Omega_0 \to \Omega(t)$ is a C^3-diffeomorphism that satisfies $\frac{d}{dt}\Phi_t^0(\cdot) = \mathbf{w}(t, \Phi_t^0(\cdot))$ and $\Phi_0^0(\cdot) = \mathrm{Id}(\cdot)$ for a given C^2 velocity field $\mathbf{w}\colon [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, which we assume satisfies the uniform bound $|\nabla_{\Omega(t)} \cdot \mathbf{w}(t)| \leq C$ for all $t \in [0,T]$. A C^2 normal vector field on the hypersurfaces is denoted by $\nu\colon [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. It follows that the Jacobian $J_t^0 := \det \mathbf{D}\Phi_t^0$ is C^2 and is uniformly bounded away from zero and infinity.

For $u\colon \Omega_0 \to \mathbb{R}$ and $v\colon \Omega(t) \to \mathbb{R}$, define the pushforward $\phi_t u = u \circ \Phi_0^t$ and pullback $\phi_{-t} v = v \circ \Phi_t^0$, where $\Phi_0^t := (\Phi_t^0)^{-1}$. We showed in §2.4.1 that $\phi_t\colon L^2(\Omega_0) \to L^2(\Omega(t))$ and $\phi_t\colon H^1(\Omega_0) \to H^1(\Omega(t))$ are linear homeomorphisms (with uniform bounds) and (thus) with $L^2 \equiv \{L^2(\Omega(t))\}_{t\in[0,T]}$, $H^1 \equiv \{H^1(\Omega(t))\}_{t\in[0,T]}$ and $H^{-1} \equiv \{H^{-1}(\Omega(t))\}_{t\in[0,T]}$, the spaces $L^2_{L^2}$, $L^2_{H^1}$ and $L^2_{H^{-1}}$ are well-defined (see §2.2 and [49] for an overview of Lebesgue and Sobolev spaces on hypersurfaces) and we let $L^2_{H^1} \subset L^2_{L^2} \subset L^2_{H^{-1}}$ be a Gelfand triple.

Define the Hilbert spaces (see Chapters 1 and 2)

$$\begin{aligned} \mathcal{W}(H^1(\Omega_0), H^{-1}(\Omega_0)) &= \{u \in L^2(0,T;H^1(\Omega_0)) \mid u' \in L^2(0,T;H^{-1}(\Omega_0))\} \\ W(H^1, H^{-1}) &= \{u \in L^2_{H^1} \mid \dot{u} \in L^2_{H^{-1}}\} \end{aligned}$$

endowed with the natural inner products. For subspaces $X \hookrightarrow H^1$ and $Y \hookrightarrow H^{-1}$, we also define the subset $W(X,Y) \subset W(H^1,H^{-1})$ in the natural manner.

Some useful results

In this subsection, p and q are not necessarily conjugate. The first part of the following lemma is a particular realisation of Lemma 1.2.23.

Lemma 3.2.1. For $p, q \in [1,\infty]$, the spaces $L^p_{L^q}$ and $L^p(0,T;L^q(\Omega_0))$ are isomorphic via the map $\phi_{(\cdot)}$ with an equivalence of norms. If $q = \infty$ the spaces are isometrically isomorphic. The embedding $L^\infty_{L^\infty} \subset L^p_{L^q}$ is continuous.

Lemma 3.2.2. The space $W(H^1, H^{-1})$ is compactly embedded in $L^2_{L^2}$.

Theorem 3.2.3 (Dominated convergence theorem for $L^p_{L^q}$)**.** Let p, $q \in [1, \infty)$. Let $\{w_n\}$ and w be functions such that $\{\tilde{w}_n\}$ and $\tilde{w}$ are measurable (eg. membership of $L^1_{L^1}$ will suffice). If for almost all $t \in [0, T]$,

$$\begin{aligned} w_n(t) \to w(t) &\quad \text{almost everywhere in } \Omega(t) \\ \exists g \in L^p_{L^q} : |w_n(t)| \leq g(t) &\quad \text{almost everywhere in } \Omega(t) \text{ and for all } n, \end{aligned}$$

then $w_n \to w$ in $L^p_{L^q}$.

Lemma 3.2.4. If $u \in W(H^1, H^{-1})$, then

$$\begin{aligned} &2\int_0^T \langle \dot{u}(t), u^+(t)\rangle_{H^{-1}(\Omega(t)),H^1(\Omega(t))} \\ &\quad = \int_{\Omega(T)} u^+(T)^2 - \int_{\Omega_0} u^+(0)^2 - \int_0^T \int_{\Omega(t)} u^+(t)^2 \nabla_\Omega \cdot \mathbf{w} \end{aligned} \tag{3.5}$$

where $u^+ := \max(u, 0)$.

Proof. By density, we can find $\{u_n\} \subset W(H^1, L^2)$ with $u_n \to u$ in $W(H^1, H^{-1})$. It follows that $\partial^\bullet(u_n^+) = \dot{u}_n \chi_{u_n \geq 0} \in L^2_{L^2}$ (this is sensible because $w \in H^1(\Omega)$ implies $w^+ \in H^1(\Omega)$ [32, Example 2.89]) and therefore (3.5) holds for u_n. Since $W(H^1, H^{-1}) \hookrightarrow C^0_{L^2}$, it follows that $u_n^+(t) \to u^+(t)$ in $L^2(\Omega(t))$ (for example see [32, Lemma 2.88] or [74, Lemma 1.22]). So we can pass to the limit in the first two terms on the right hand side.

Now we just need to show that $u_n^+ \to u^+$ in $L^2_{H^1}$. It is easy to show the convergence in $L^2_{L^2}$, so we need only to check the convergence of the gradient. Let $g(r) = \chi_{\{r>0\}}$. Then, using $g \leq 1$,

$$\begin{aligned} |\nabla_\Omega u_n^+(t,x) - \nabla_\Omega u^+(t,x)| &\leq |\nabla_\Omega u_n(t,x) - \nabla_\Omega u(t,x)| \\ &\quad + |g(u_n(t,x)) - g(u(t,x))||\nabla_\Omega u(t,x)|. \end{aligned}$$

For the second term, let us note that since $u_n \to u$ in $L^2_{H^1}$, for almost all t, $u_n(t,x) \to u(t,x)$ almost everywhere in $\Omega(t)$ for a subsequence (which we have not relabelled). Let us fix t. Then for almost every $x \in \Omega(t)$, it follows that $g(u_n(t,x))\nabla_\Omega u(t,x) \to g(u(t,x))\nabla_\Omega u(t,x)$ pointwise. Because $g \leq 1$, the dominated convergence theorem gives overall $\nabla_\Omega u_n^+ \to \nabla_\Omega u^+$ in $L^2_{L^2}$. □

3.2.2 Preliminary results

Remark 3.2.5. It is not clear, *a priori*, that the solution of a Stefan problem will separate the domain into a region where the temperature is (strictly) positive and a region where it is (strictly) negative *with an interface that has measure zero*. The interface may not even be connected. If the interior of the set where the temperature is zero is non-empty, then that interior is called a *mushy region*. The mushy region is filled with the substance in a state that is neither solid nor liquid.

It is well known in the standard setting that a mushy region can arise in the presence of heat sources [12, 54]; even without any heat sources, certain initial data may give rise to mushy regions. We will content ourselves with the following heuristic calculations under the assumption that there is no mushy region.

Let the bounded weak solution of (3.1) (in the sense of Definition 3.1.2) have the additional regularity $u \in W(H^1, L^2)$ and $\Delta_\Omega u \in L^2_{L^2}$, and suppose that the sets $\Omega_l(t) = \{u > 0\}$ and $\Omega_s(t) = \{u < 0\}$ divide $\Omega(t)$ with a common interface $\Gamma(t)$, which we assume is a sufficiently smooth $n-$dimensional hypersurface (of measure zero with respect to the surface measure on $\Omega(t)$). Then the bounded weak solution is also a classical solution in the sense of (3.3). To see this, suppose that (u, e) is a weak solution satisfying the equality in (3.4). The integration by parts formula on each subdomain of Ω implies

$$\begin{aligned}\int_0^T \int_{\Omega(t)} \nabla_\Omega u(t) \nabla_\Omega \eta(t) = &-\int_0^T \int_{\Omega(t)} \eta(t) \Delta_\Omega u(t) \\ &+ \int_0^T \int_{\Gamma(t)} \eta(t)(\nabla_\Omega u_s(t) - \nabla_\Omega u_l(t)) \cdot \mu. \end{aligned} \tag{3.6}$$

With $e(t)\eta(t)\nabla_\Omega \cdot \mathbf{w} = \nabla_\Omega \cdot (e(t)\eta(t)\mathbf{w}) - \mathbf{w} \cdot \nabla_\Omega(e(t)\eta(t))$ and the divergence theorem [49, §2.2],

$$\begin{aligned}\int_0^T \int_{\Omega_s(t)} e(t)\eta(t)\nabla_\Omega \cdot \mathbf{w} = &\int_0^T \int_{\Gamma(t)} e(t)\eta(t)\mathbf{w} \cdot \mu \\ &+ \int_0^T \int_{\Omega_s(t)} \mathbf{w} \cdot (e(t)\eta(t)\nu H - \nabla_\Omega(e(t)\eta(t))).\end{aligned}$$

We use this result in the formula for integration by parts over time over Ω_s:

$$\begin{aligned}\int_0^T \int_{\Omega_s(t)} \dot{\eta}(t) e(t) = &\int_0^T \frac{d}{dt} \int_{\Omega_s(t)} e(t)\eta(t) - \int_0^T \int_{\Omega_s(t)} \dot{e}(t)\eta(t) \\ &- \int_0^T \int_{\Gamma(t)} e(t)\eta(t)\mathbf{w} \cdot \mu - \int_0^T \int_{\Omega_s(t)} e(t)\eta(t)\mathbf{w} \cdot \nu H \\ &+ \int_0^T \int_{\Omega_s(t)} \mathbf{w} \cdot \nabla_\Omega(e(t)\eta(t)).\end{aligned}$$

A similar expression over Ω_l can also be derived this way, the difference being that the term with μ has the opposite sign. Then, using $\dot{e} = \partial^\bullet(\mathcal{E}(u)) = \dot{u}$, $e_s(t)|_{\Gamma(t)} = 0$, and $e_l(t)|_{\Gamma(t)} = 1$, we get

$$\int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) = \int_0^T \frac{d}{dt}\int_{\Omega(t)} e(t)\eta(t) - \int_0^T \int_{\Omega(t)} \dot{u}(t)\eta(t) + \int_0^T \int_{\Gamma(t)} \eta(t)\mathbf{w}\cdot\mu$$
$$- \int_0^T \int_{\Omega(t)} e(t)\eta(t)\mathbf{w}\cdot\nu H + \int_0^T \int_{\Omega(t)} \mathbf{w}\cdot\nabla_\Omega(e(t)\eta(t)). \quad (3.7)$$

Since by the partial integration formula $\int_{\Omega(t)} \underline{D}_i(g) = \int_{\Omega(t)} gH\nu_i$, we have (with $g = \mathbf{w}_i e(t)\eta(t)$) that the fourth term in the right hand side of (3.7) is

$$\int_{\Omega(t)} e(t)\eta(t)\mathbf{w}\cdot\nu H = \sum_i \int_{\Omega(t)} e(t)\eta(t)\mathbf{w}_i\nu_i H$$
$$= \int_{\Omega(t)} \nabla_\Omega(e(t)\eta(t))\cdot\mathbf{w} + \int_{\Omega(t)} \eta(t)e(t)\nabla_\Omega\cdot\mathbf{w}.$$

So the calculation (3.7) becomes

$$\int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t)$$
$$= \int_0^T \left(\frac{d}{dt}\int_{\Omega(t)} e(t)\eta(t) - \int_{\Omega(t)} (\dot{u}(t)\eta(t) + \eta(t)e(t)\nabla_\Omega\cdot\mathbf{w}) + \int_{\Gamma(t)} \eta(t)\mathbf{w}\cdot\mu\right). \quad (3.8)$$

Now, taking the weak formulation (3.4) and substituting (3.8) together with the expression for the spatial term (3.6), we get for η with $\eta(T) = \eta(0) = 0$

$$\int_0^T \int_{\Omega(t)} f(t)\eta(t) = -\int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) + \int_0^T \int_{\Omega(t)} \nabla_\Omega u(t)\nabla_\Omega\eta(t)$$
$$= \int_0^T \int_{\Omega(t)} (\dot{u}(t) + e(t)\nabla_\Omega\cdot\mathbf{w} - \Delta_\Omega u(t))\eta(t)$$
$$+ \int_{\Gamma(t)} \eta(t)\left((\nabla_\Omega u_s(t) - \nabla_\Omega u_l(t))\cdot\mu - (\mathbf{w}\cdot\mu)\right).$$

Taking η to be compactly supported in Q_s, and afterwards taking η compactly supported in Q_l, we recover exactly the first two equations in (3.3). So we may drop the first integral on the left and the right hand side. Then with a careful choice of η, we will obtain precisely the interface condition in (3.3).

Before moving on, let us state a well-posedness result for a linear PDE that will be helpful when we prove continuous dependence (it is the "dual equation" that arises there) in §3.3.3.

Lemma 3.2.6. Given $\xi \in C^1(\Omega_0)$ and $\tilde{\alpha} \in C^2([0,T]\times\Omega_0)$ satisfying $0 < \epsilon \le \alpha \le \alpha_0$ a.e., there exists a unique solution $\varphi \in W(H^1, L^2)$ with $\Delta_\Omega \varphi \in L^2_{L^2}$ to

$$\begin{aligned} \dot{\varphi} - \alpha(x,t)\Delta_\Omega \varphi = 0 \\ \varphi(x,0) = \xi(x) \end{aligned} \tag{3.9}$$

satisfying $\|\varphi\|_{L^\infty_{L^\infty}} \le \|\xi\|_{L^\infty(\Omega_0)}$ and (cf. [81, Chapter V, §9])

$$\begin{aligned} &\int_0^t \int_{\Omega(\tau)} (\dot{\varphi}(\tau))^2 + \int_0^t \int_{\Omega(\tau)} \alpha|\Delta_\Omega\varphi|^2 + \int_{\Omega(t)} |\nabla_\Omega \varphi(t)|^2 \\ &\quad \le (1+\alpha_0)(1+e^{2C_{\mathbf{w}}(1+\alpha_0)t}) \int_{\Omega_0} |\nabla_\Omega \xi|^2. \end{aligned} \tag{3.10}$$

Proof. Define the bilinear form $a(t;\varphi,\eta) = \int_{\Omega(t)} \alpha(x,t)\nabla_\Omega\varphi\nabla_\Omega\eta + \nabla_\Omega\alpha(x,t)\nabla_\Omega\varphi\eta$ which is clearly bounded and coercive on $H^1(\Omega(t))$. Split $a(t;\cdot,\cdot)$ into the forms $a_s(t;\varphi,\eta) := \int_{\Omega(t)} \alpha(x,t)\nabla_\Omega\varphi\nabla_\Omega\eta$ and $a_n(t;\varphi,\eta) := \int_{\Omega(t)} \nabla_\Omega\alpha(x,t)\nabla_\Omega\varphi\eta$. One sees that $a_s(t;\eta,\eta) \ge 0$ and that both $a_n(t;\cdot,\cdot)\colon H^1(\Omega(t)) \times L^2(\Omega(t)) \to \mathbb{R}$ and $a_s(t;\cdot,\cdot)\colon H^1(\Omega(t)) \times H^1(\Omega(t)) \to \mathbb{R}$ are bounded. Also, letting $\chi_j^t := \phi_t\chi_j^0$ where χ_j^0 are the normalised eigenfunctions of $-\Delta_{\Omega_0}$, we have for $\eta \in \tilde{C}^1_{H^1} := \{u \mid u(t) = \sum_{j=1}^m \alpha_j(t)\chi_j^t,\ m \in \mathbb{N},\ \alpha_j \in AC([0,T]) \text{ and } \alpha_j' \in L^2(0,T)\}$,

$$\frac{d}{dt} a_s(t;\eta(t),\eta(t)) = 2a_s(t;\dot{\eta}(t),\eta(t)) + r(t;\eta(t))$$

where r is such that $|r(t;\eta(t))| \le C\,\|\eta(t)\|^2_{H^1(\Omega(t))}$ (see [49, Lemma 2.1], note that $\tilde{\alpha} \in C^1([0,T];C^1(\Omega_0))$ and thus $\alpha \in C^1_{H^1}$). Hence by Theorem 1.4.8 we have the unique existence of $\varphi \in W(H^1,L^2)$. Rearranging the equation (3.9) shows that $\alpha\Delta_\Omega\varphi \in L^2_{L^2}$. Since α is uniformly bounded by positive constants, it follows that $\Delta_\Omega\varphi \in L^2_{L^2}$.

The L^∞ bound Let $K := \|\xi\|_{L^\infty(\Omega_0)}$. Test the equation with $(\varphi - K)^+$:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|(\varphi(t)-K)^+\right\|^2_{L^2(\Omega(t))} + \int_{\Omega(t)} \alpha(t)\nabla_\Omega((\varphi(t)-K)^+)\nabla_\Omega\varphi(t) \\ &\quad = \frac{1}{2}\int_{\Omega(t)} ((\varphi(t)-K)^+)^2 \nabla_\Omega\cdot\mathbf{w} - \int_{\Omega(t)} \nabla_\Omega\alpha(t)\nabla_\Omega\varphi(t)(\varphi(t)-K)^+ \end{aligned}$$

which becomes, through the use of Young's inequality with δ,

$$\frac{1}{2}\frac{d}{dt}\left\|(\varphi(t)-K)^{+}\right\|_{L^2(\Omega(t))}^2 \leq \left(\frac{C_{\mathbf{w}}}{2}+\|\nabla_\Omega \alpha\|_{L^\infty} C_\delta\right)\left\|(\varphi(t)-K)^{+}\right\|_{L^2(\Omega(t))}^2.$$

An application of Gronwall's inequality and noticing $(\varphi(0)-K)^+ = (\xi - \|\xi\|_{L^\infty})^+ = 0$ yields $\varphi(t) \leq \|\xi\|_{L^\infty(\Omega_0)}$. Repeating this process with $(-\varphi(t)-K)^+$ allows us to conclude.

The inequality (3.10) Multiplying the equation (3.9) by $\Delta_\Omega\varphi$ and integrate: formally,

$$\begin{aligned}
\int_0^t\int_{\Omega(\tau)}\alpha|\Delta_\Omega\varphi|^2 &= -\int_0^t\int_{\Omega(\tau)}\nabla_\Omega\dot{\varphi}\nabla_\Omega\varphi \\
&= -\int_0^t\frac{1}{2}\frac{d}{d\tau}\int_{\Omega(\tau)}|\nabla_\Omega\varphi|^2 + \frac{1}{2}\int_0^t\int_{\Omega(\tau)}|\nabla_\Omega\varphi|^2\nabla_\Omega\cdot\mathbf{w} \\
&\quad - \int_0^t\int_{\Omega(\tau)}D(\mathbf{w})\nabla_\Omega\varphi\nabla_\Omega\varphi \\
&\leq \frac{1}{2}\int_{\Omega_0}|\nabla_\Omega\xi|^2 - \frac{1}{2}\int_{\Omega(t)}|\nabla_\Omega\varphi(t)|^2 + C_{\mathbf{w}}\int_0^t\int_{\Omega(\tau)}|\nabla_\Omega\varphi|^2. \qquad (3.11)
\end{aligned}$$

See [49, Lemma 2.1] or §2.5 for the definition of the matrix $D(\mathbf{w})$. This calculation is merely formal because we have not shown that $\dot{\varphi}(t) \in H^1(\Omega(t))$; however the end result of the calculation is still valid by Lemma 3.2.7. We also have by squaring (3.9), integrating and using (3.11):

$$\int_0^t\int_{\Omega(\tau)}(\dot{\varphi}(\tau))^2 \leq \alpha_0\int_0^t\int_{\Omega(\tau)}\alpha(\Delta_\Omega\varphi)^2 \leq \frac{\alpha_0}{2}\int_{\Omega_0}|\nabla_\Omega\xi|^2 + \alpha_0 C_{\mathbf{w}}\int_0^t\int_{\Omega(\tau)}|\nabla_\Omega\varphi|^2.$$

Adding the last two inequalities then we obtain

$$\begin{aligned}
\int_0^t\int_{\Omega(\tau)}(\dot{\varphi}(\tau))^2 + &\int_0^t\int_{\Omega(\tau)}\alpha|\Delta_\Omega\varphi|^2 + \frac{1}{2}\int_{\Omega(t)}|\nabla_\Omega\varphi(t)|^2 \\
&\leq \frac{1+\alpha_0}{2}\int_{\Omega_0}|\nabla_\Omega\xi|^2 + C_{\mathbf{w}}(1+\alpha_0)\int_0^t\int_{\Omega(\tau)}|\nabla_\Omega\varphi|^2.
\end{aligned}$$

Gronwall's inequality can be used to deal with the last term on the right hand side. □

In the next lemma, we rigorously justify (3.11).

Lemma 3.2.7. With $\varphi \in W(H^1, L^2)$ from the previous lemma, the following inequality holds:

$$\int_0^t \int_{\Omega(\tau)} \alpha |\Delta_\Omega \varphi|^2 \leq \frac{1}{2}\int_{\Omega_0} |\nabla_\Omega \xi|^2 - \frac{1}{2}\int_{\Omega(t)} |\nabla_\Omega \varphi(t)|^2 + C_{\mathbf{w}} \int_0^t \int_{\Omega(\tau)} |\nabla_\Omega \varphi|^2. \tag{3.12}$$

Proof. Let $C^\infty_{H^2} := \{\eta \mid \phi_{-(\cdot)}\eta(\cdot) \in C^\infty([0,T]; H^2(\Omega_0))\}$. We start with a few preliminary results. Let us show $C^\infty_{H^2} \subset W(H^2, H^1)$. Take $\eta \in C^\infty_{H^2}$ so that $\tilde{\eta} \in C^\infty([0,T]; H^2(\Omega_0)) \subset \mathcal{W}(H^2, H^1)$. By smoothness of $\Phi^{(\cdot)}_0$, it follows that $\eta = \phi_{(\cdot)}\tilde{\eta} \in L^2_{H^2}$, and $\dot{\eta} = \partial^\bullet(\phi_{(\cdot)}\tilde{\eta}) = \phi_{(\cdot)}(\tilde{\eta}') \in L^2_{H^1}$ because $\tilde{\eta}' \in C^\infty([0,T]; H^2(\Omega_0)) \subset L^2(0,T; H^1(\Omega_0))$. So $\eta \in W(H^2, H^1)$.

Let us also prove that $C^\infty_{H^2} \subset W(H^2, L^2)$ is dense. Let $w \in W(H^2, L^2)$; then $\tilde{w} \in \mathcal{W}(H^2, L^2)$ since $\tilde{w} \in L^2(0,T; H^2(\Omega_0))$ by smoothness of $\Phi^{(\cdot)}_0$ and since $\tilde{w}' = \phi_{-(\cdot)}\dot{w} \in L^2(0,T; L^2(\Omega_0))$ (because $\dot{w} \in L^2_{L^2}$). By [24, Lemma II.5.10] there exists $\tilde{w}_n \in C^\infty([0,T]; H^2(\Omega_0))$ with $\tilde{w}_n \to \tilde{w}$ in $\mathcal{W}(H^2, L^2)$. Then, $w_n := \phi_{(\cdot)}\tilde{w}_n \in C^\infty_{H^2}$ (by definition) and

$$\|w_n - w\|_{W(H^2,L^2)} \leq C\left(\|\tilde{w}_n - \tilde{w}\|_{L^2(0,T;H^2(\Omega_0))} + \left\|\tilde{w}'_n - \tilde{w}'\right\|_{L^2(0,T;L^2(\Omega_0))}\right) \to 0,$$

where we used the smoothness of $\Phi^{(\cdot)}_0$ and the reasoning behind Assumption 1.2.50 (see also Theorem 1.2.46).

Given $\varphi \in W(H^2, L^2)$, by the density result, there exists $\varphi_n \in C^\infty_{H^2} \subset W(H^2, H^1)$ such that $\varphi_n \to \varphi$ in $W(H^2, L^2)$ with φ_n satisfying (3.12):

$$\begin{aligned}\int_0^t \int_{\Omega(\tau)} \alpha |\Delta_\Omega \varphi_n|^2 \leq \frac{1}{2}\int_{\Omega_0} |\nabla_\Omega \varphi_n(0)|^2 - \frac{1}{2}\int_{\Omega(t)} |\nabla_\Omega \varphi_n(t)|^2 \\ + C_{\mathbf{w}} \int_0^t \int_{\Omega(\tau)} |\nabla_\Omega \varphi_n|^2.\end{aligned} \tag{3.13}$$

We know that $\tilde{\varphi}_n \to \tilde{\varphi}$ in $\mathcal{W}(H^2, L^2)$ (this is just how we construct the sequence φ_n; see above), and $\mathcal{W}(H^2, L^2) \hookrightarrow C^0([0,T]; H^1(\Omega_0))$ [24, Lemma II.5.14] implies $\varphi_n(t) \to \varphi(t)$ in $H^1(\Omega(t))$. Now we can pass to the limit in every term in (3.13). $\square$

3.3 Well-posedness

We can approximate $\mathcal{E}$ (see (3.2)) by C^∞ bi-Lipschitz functions $\mathcal{E}_\epsilon$ such that (for example see [106, 105])

$$\begin{aligned}
&\mathcal{E}_\epsilon \to \mathcal{E} \text{ uniformly in the compact subsets of } \mathbb{R}\backslash\{0\}\\
&\mathcal{E}_\epsilon^{-1} \to \mathcal{E}^{-1} \text{ uniformly in the compact subsets of } \mathbb{R}\\
&\mathcal{E}_\epsilon(0) = 0 \text{ and } \mathcal{E}_\epsilon = \mathcal{E} \text{ on } (-\infty, 0) \cup (\epsilon, \infty)\\
&1 \le \mathcal{E}_\epsilon'(r) \le 1 + L_\epsilon \text{ and } (1+L_\epsilon)^{-1} \le (\mathcal{E}_\epsilon^{-1}(r))' \le 1 \quad \text{for all } r \in \mathbb{R}
\end{aligned}$$

(where $L_\epsilon = \mathcal{O}(1/\epsilon)$ is the Lipschitz constant of the approximation to the Heaviside function). We write $\mathcal{U} := \mathcal{E}^{-1}$ and $\mathcal{U}_\epsilon := \mathcal{E}_\epsilon^{-1}$. In order to prove Theorem 3.1.3, that of the well-posedness of bounded weak solutions given bounded data, we consider the following approximation of (3.1).

Definition 3.3.1. Find for each $\epsilon > 0$ a function $e_\epsilon \in W(H^1, H^{-1})$ such that

$$\begin{aligned}
\partial^\bullet e_\epsilon - \Delta_\Omega(\mathcal{U}_\epsilon e_\epsilon) + e_\epsilon \nabla_\Omega \cdot \mathbf{w} &= f \quad \text{in } L^2_{H^{-1}}\\
e_\epsilon(0) &= e_0.
\end{aligned} \tag{$\mathbf{P}_\epsilon$}$$

To prove existence for this problem, we will use a fixed point theorem along with the linear theory in Chapter 1.

Theorem 3.3.2. Given $f \in L^2_{H^{-1}}$ and $e_0 \in L^2(\Omega_0)$, the problem $(\mathbf{P}_\epsilon)$ has a weak solution $e_\epsilon \in W(H^1, H^{-1})$.

Proof. Using the chain rule on the nonlinear term leads us to consider for fixed $w \in W(H^1, H^{-1})$

$$\begin{aligned}
\langle \partial^\bullet(Sw), \eta\rangle_{L^2_{H^{-1}}, L^2_{H^1}} + (\mathcal{U}_\epsilon'(w)\nabla_\Omega(Sw), \nabla_\Omega \eta)_{L^2_{L^2}} + (Sw, \eta\nabla_\Omega \cdot \mathbf{w})_{L^2_{L^2}}&\\
= \langle f, \eta\rangle_{L^2_{H^{-1}}, L^2_{H^1}}&\\
Sw(0) = e_0.&
\end{aligned} \tag{$\mathbf{P}(w)$}$$

If S denotes the solution map of $(\mathbf{P}(w))$ that takes $w \mapsto Sw$, then we seek a fixed point of S. First, note that since the bilinear form involving the surface gradients is bounded and coercive, the solution $Sw \in W(H^1, H^{-1})$ of $(\mathbf{P}(w))$ does indeed exist by Theorem 1.4.1, and moreover, it satisfies the estimate

$$\|Sw\|_{W(H^1,H^{-1})} \le C\left(\|f\|_{L^2_{H^{-1}}} + \|u_0\|_{L^2(\Omega_0)}\right) =: C_* \tag{3.14}$$

where the constant C does not depend on w because $\mathcal{U}'_\epsilon(w(t))$ is uniformly bounded from below (in w). Then the set

$$E := \{w \in W(H^1, H^{-1}) \mid w(0) = e_0, \|w\|_{W(H^1,H^{-1})} \leq C_*\},$$

which is a closed, convex, and bounded subset of $X := W(H^1, H^{-1})$, is such that $S(E) \subset E$ by (3.14). We now show that S is weakly continuous. Let $w_n \rightharpoonup w$ in $W(H^1, H^{-1})$ with $w_n \in E$. From the estimate (3.14), we know that Sw_n is bounded in $W(H^1, H^{-1})$, so for a subsequence

$$\begin{aligned} Sw_{n_j} &\rightharpoonup \chi \quad \text{in } W(H^1, H^{-1}) \\ Sw_{n_j} &\to \chi \quad \text{in } L^2_{L^2} \end{aligned}$$

by the compact embedding of Lemma 3.2.2. Now we show that $\chi = Sw$. Due to $W(H^1, H^{-1}) \hookrightarrow C^0_{L^2}$, $Sw_{n_j} \rightharpoonup \chi$ in $C^0_{L^2}$. This implies $Sw_{n_j}(0) \rightharpoonup \chi(0)$ in $L^2(\Omega_0)$ (to see this consider for arbitrary $f \in L^2(\Omega_0)$ the functional $G \in (C^0_{L^2})^*$ defined $G(u_n) = \int_{\Omega_0} f u_n(0)$). Since $Sw_{n_j}(0) = e_0$, it follows that

$$\chi(0) = e_0. \tag{3.15}$$

On the other hand, since w_n are weakly convergent in $W^1(H^1, H^{-1})$, they are bounded in the same space. Now, $W(H^1, H^{-1}) \overset{c}{\hookrightarrow} L^2_{L^2}$, hence $w_n \to w$ in $L^2_{L^2}$. It follows that the subsequence $w_{n_j} \to w$ in $L^2_{L^2}$ too, and so there is a subsequence such that for almost every $t \in [0, T]$, $w_{n_{j_k}}(t) \to w(t)$ a.e. in $\Omega(t)$. By continuity, for a.a. t, $\mathcal{U}'_\epsilon(w_{n_{j_k}}(t))\nabla_\Omega \eta(t) \to \mathcal{U}'_\epsilon(w(t))\nabla_\Omega \eta(t)$ a.e., and also we have $|\mathcal{U}'_\epsilon(w_{n_{j_k}})\nabla_\Omega \eta| \leq |\nabla_\Omega \eta|$ with the right hand side in $L^2_{L^2}$. Thus we can use the dominated convergence theorem (Theorem 3.2.3) which tells us that $\mathcal{U}'_\epsilon(w_{n_{j_k}})\nabla_\Omega \eta \to \mathcal{U}'_\epsilon(w)\nabla_\Omega \eta$ in $L^2_{L^2}$. Now we pass to the limit in the equation $(\mathbf{P}(w))$ with w replaced by $w_{n_{j_k}}$ to get

$$\int_0^T \langle \partial^\bullet \chi(t), \eta(t) \rangle + \int_{\Omega(t)} \mathcal{U}'_\epsilon(w(t))\nabla_\Omega \chi(t) \nabla_\Omega \eta(t) + \chi(t)\eta(t)\nabla_\Omega \cdot \mathbf{w} = \int_0^T \langle f(t), \eta(t) \rangle$$

which, along with (3.15), shows that $\chi = Sw$, so $Sw_{n_j} \rightharpoonup Sw$. However, for the weak continuity, we have to show that the whole sequence converges, not just a subsequence. Let $x_n = Sw_n$ and equip the space $X = W(H^1, H^{-1})$ with the weak topology. Let $x_{n_m} = Sw_{n_m}$ be a subsequence. By the bound of S, it follows that x_{n_m} is bounded, hence it has a subsequence such that

$$x_{n_{m_l}} \rightharpoonup x^* \text{ in } X \qquad \text{and} \qquad x_{n_{m_l}} \to x^* \text{ in } L^2_{L^2}.$$

By similar reasoning as before, we identify $x^* = Sw$, and Theorem 3.3.3 below tells us that indeed $x_n = Sw_n \rightharpoonup Sw$. Then by the Schauder–Tikhonov fixed point theorem [60, Theorem 1.4, p. 118], S has a fixed point. $\square$

Theorem 3.3.3. Let x_n be a sequence in a topological space X such that every subsequence x_{n_j} has a subsequence $x_{n_{j_k}}$ converging to $x \in X$. Then the full sequence x_n converges to x.

3.3.1 Uniform estimates

We set $u_\epsilon = \mathcal{U}_\epsilon(e_\epsilon)$. Below we denote by M a constant such that $\|u_0\|_{L^\infty(\Omega_0)} \leq M$. We now obtain various estimates uniform in ϵ in order to pass to the limit.

Lemma 3.3.4. The following bound holds independent of ϵ:

$$\|u_\epsilon\|_{L^\infty_{L^\infty}} + \|\mathcal{E}_\epsilon(u_\epsilon)\|_{L^\infty_{L^\infty}} \leq 2e^{\|\nabla_\Omega \cdot \mathbf{w}\|_\infty T}\left(T\|f\|_{L^\infty_{L^\infty}} + \|u_0\|_{L^\infty(\Omega_0)} + 1\right) + 1.$$

Proof. We substitute $w(t) = e^{-\lambda t}e_\epsilon(t)$ in $(\mathbf{P}_\epsilon)$ and use $\partial^\bullet(e^{\lambda t}w(t)) = \lambda e^{\lambda t}w(t) + e^{\lambda t}\dot{w}(t)$ to get

$$\dot{w}(t) - e^{-\lambda t}\Delta_\Omega(\mathcal{U}^\epsilon(e^{\lambda t}w(t))) + \lambda w(t) + w(t)\nabla_\Omega \cdot \mathbf{w} = e^{-\lambda t}f(t).$$

Let $\alpha = \|f\|_{L^\infty_{L^\infty}}$ and $\beta = \|e_0\|_{L^\infty(\Omega_0)}$ and define $v(t) = \alpha t + \beta$. Note that $\dot{v}(t) = \alpha$ and $v(0) = \beta$. Subtracting $\dot{v}(t)$ from the above and testing with $(w(t) - v(t))^+$, we get

$$\begin{aligned} &\langle \dot{w}(t) - \dot{v}(t), (w(t) - v(t))^+\rangle + \int_{\Omega(t)} e^{-\lambda t}\nabla_\Omega(\mathcal{U}^\epsilon(e^{\lambda t}w(t)))\nabla_\Omega(w(t) - v(t))^+ \\ &+ \int_{\Omega(t)} (\lambda + \nabla_\Omega \cdot \mathbf{w})w(t)(w(t) - v(t))^+ = \int_{\Omega(t)} (e^{-\lambda t}f(t) - \alpha)(w(t) - v(t))^+ \end{aligned} \tag{3.16}$$

where of course the duality pairing is between $H^{-1}(\Omega(t))$ and $H^1(\Omega(t))$. Note that $e^{-\lambda t}\nabla_\Omega(\mathcal{U}_\epsilon(e^{\lambda t}w(t)))\nabla_\Omega(w(t) - v(t))^+ = \mathcal{U}_\epsilon'(e^{\lambda t}w(t))|\nabla_\Omega(w(t) - v(t))^+|^2$ because $\nabla_\Omega v(t) = 0$. Set $\lambda := \|\nabla_\Omega \cdot \mathbf{w}\|_{L^\infty}$, then the last term on the LHS of (3.16) is non-negative because if $w > v$, $w > 0$ since $v \geq 0$. So we can disregard that and the gradient term to find

$$\langle \dot{w}(t) - \dot{v}(t), (w(t) - v(t))^+\rangle_{H^{-1}(\Omega(t)),H^1(\Omega(t))} \leq \int_{\Omega(t)} (e^{-\lambda t}f(t) - \alpha)(w(t) - v(t))^+.$$

Integrating this and using Lemma 3.2.4, we find

$$\frac{1}{2}\int_{\Omega(T)}((w(t)-v(t))^+)^2 \le \frac{1}{2}\|\nabla_\Omega\cdot\mathbf{w}\|\int_0^T\int_{\Omega(t)}((w(t)-v(t))^+)^2$$

as $e^{-\lambda t}f(t)-\alpha = e^{-\lambda t}f(t)-\|f(t)\|_{L^\infty(\Omega(t))}\le 0$ and $w(0)-v(0)=e_0-\|e_0\|_{L^\infty(\Omega_0)}\le 0$. The use of Gronwall's inequality gives $w(t)\le T\|f\|_{L^\infty_{L^\infty}}+(1+M)$ almost everywhere on $\Omega(t)$. So we have shown that for all $t\in[0,T]\backslash N_1$, $w(t,x)\le C$ for all $x\in\Omega(t)\backslash M_1^t$, where $\mu(N_1)=\mu(M_1^t)=0$. A similar argument yields for all $t\in[0,T]\backslash N_2$, $w(t,x)\ge -C$ for all $x\in\Omega(t)\backslash M_2^t$, where $\mu(N_2)=\mu(M_2^t)=0$. Taking these statements together tells us that for all $t\in[0,T]\backslash N$, $|w(t,x)|\le C$ on $\Omega(t)\backslash M^t$ where $N=N_1\cup N_2$ and $M^t=M_1^t\cup M_2^t$ have measure zero. This gives $\|w\|_{L^\infty_{L^\infty}}\le T\|f\|_{L^\infty_{L^\infty}}+(1+M)$. From this and $u_\epsilon=\mathcal{U}_\epsilon(e^{\lambda(\cdot)}w(\cdot))\le e^{\lambda T}|w|$, we obtain the bound on u_ϵ. The bound on $\mathcal{E}_\epsilon(u_\epsilon)$ follows from $\mathcal{E}_\epsilon(u_\epsilon)\le 1+|u_\epsilon|$. □

Lemma 3.3.5. The following bound holds independent of ϵ:

$$\|\nabla_\Omega u_\epsilon\|_{L^2_{L^2}}+\|\partial^\bullet(\mathcal{E}_\epsilon u_\epsilon)\|_{L^2_{H^{-1}}}\le C(T,\Omega,M,\mathbf{w},f). \tag{3.17}$$

Proof. Testing with $\mathcal{E}_\epsilon(u_\epsilon)$ in $(\mathbf{P}_\epsilon)$, using $\nabla_\Omega u_\epsilon\nabla_\Omega(\mathcal{E}_\epsilon(u_\epsilon))=(\mathcal{E}_\epsilon)'(u_\epsilon)|\nabla_\Omega u_\epsilon|^2\ge|\nabla_\Omega u_\epsilon|^2$, integrating over time and using the previous estimate, we find

$$\frac{1}{2}\|\mathcal{E}_\epsilon(u_\epsilon(T))\|^2_{L^2(\Omega(T))}+\int_0^T\int_{\Omega(t)}|\nabla_\Omega u_\epsilon(t)|^2\le\frac{1}{2}(1+M)^2|\Omega_0|+C_1(T,M,\mathbf{w},f).$$

The bound on the time derivative follows by taking supremums. □

These bounds are not sufficient for identifying the limit of the nonlinearity. Also, we remark that we cannot use Aubin–Lions here so we need another estimate.

Lemma 3.3.6. Define $\tilde{u}_\epsilon=\phi_{-(\cdot)}u_\epsilon$. The following limit holds uniformly in ϵ:

$$\lim_{h\to 0}\int_0^{T-h}\int_{\Omega_0}|\tilde{u}_\epsilon(t+h)-\tilde{u}_\epsilon(t)|=0.$$

Proof. We follow the proof of Theorem A.1 in [15] here. Fix $h \in (0,T)$ and consider

$$\begin{aligned}
&\int_0^{T-h} (\mathcal{E}_\epsilon(\tilde{u}_\epsilon(t+h)) - \mathcal{E}_\epsilon(\tilde{u}_\epsilon(t)), \tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t))_{L^2(\Omega_0)}\, \mathrm{d}t \\
&\quad = \int_0^{T-h} \int_t^{t+h} \frac{d}{d\tau}(\mathcal{E}_\epsilon(\tilde{u}_\epsilon(\tau)), \tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t))_{L^2(\Omega_0)}\, \mathrm{d}\tau\, \mathrm{d}t \\
&\quad \leq \sqrt{h}\left\|(\mathcal{E}_\epsilon(\tilde{u}_\epsilon))'\right\|_{L^2(0,T;H^{-1}(\Omega_0))} \int_0^{T-h} (\|\tilde{u}_\epsilon(t+h)\|_{H^1(\Omega_0)} + \|\tilde{u}_\epsilon(t)\|_{H^1(\Omega_0)})\, \mathrm{d}t \\
&\quad \leq C_1(T, \Omega, M, \mathbf{w}, f)\sqrt{h}\left\|(\mathcal{E}_\epsilon(\tilde{u}_\epsilon))'\right\|_{L^2(0,T;H^{-1}(\Omega_0))} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the uniform estimates)} \\
&\quad \leq C_2(T, \Omega, M, \mathbf{w}, f)\sqrt{h}\left\|\partial^\bullet(\mathcal{E}_\epsilon(u_\epsilon))\right\|_{L^2_{H^{-1}}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(see the proof of Theorem 1.2.46)} \\
&\quad \leq C_3(T, \Omega, M, \mathbf{w}, f)\sqrt{h}, \qquad (3.18)
\end{aligned}$$

with the last inequality by (3.17). Now, since the $\mathcal{U}_\epsilon'$ are uniformly bounded above, they are uniformly equicontinuous. Therefore, for fixed δ, there is a σ_δ (depending solely on δ) such that

$$\text{if } |y-z| < \sigma_\delta, \text{ then } |\mathcal{U}_\epsilon(y) - \mathcal{U}_\epsilon(z)| < \delta \qquad \text{for } \textit{any } \epsilon. \tag{3.19}$$

So in the set $\{|\tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t)| > \delta\} = \{|\mathcal{U}_\epsilon(\mathcal{E}_\epsilon(\tilde{u}_\epsilon(t+h))) - \mathcal{U}_\epsilon(\mathcal{E}_\epsilon(\tilde{u}_\epsilon(t)))| > \delta\}$, we must have $|\mathcal{E}_\epsilon(\tilde{u}_\epsilon(t+h)) - \mathcal{E}_\epsilon(\tilde{u}_\epsilon(t))| \geq \sigma_\delta$ (this is the contrapositive of (3.19)). This implies from (3.18) that

$$\int_0^{T-h} \int_{\Omega_0} |\tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t)|\chi_{\{|\tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t)| > \delta\}} \leq \frac{C_3\sqrt{h}}{\sigma_\delta}.$$

Writing $\mathrm{Id} = \chi_{\{|\tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t)| > \delta\}} + \chi_{\{|\tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t)| \leq \delta\}}$, notice that

$$\begin{aligned}
&\int_0^{T-h} \int_{\Omega_0} |\tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t)| \\
&\quad \leq \int_0^{T-h} \int_{\Omega_0} |\tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t)|\chi_{\{|\tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t)| > \delta\}} + \delta|\Omega_0|(T-h) \\
&\quad \leq \frac{C_3\sqrt{h}}{\sigma_\delta} + \delta|\Omega_0|T.
\end{aligned}$$

Taking the limit as $h \to 0$, using the arbitrariness of $\delta > 0$ and the fact that the right hand side of the above does not depend on ϵ gives us the result. $\square$

3.3.2 Existence of bounded weak solutions

With all the uniform estimates acquired, we can extract (weakly) convergent subsequences. In fact, we find (we have not relabelled subsequences)

$$\begin{aligned} u_\epsilon &\to u && \text{in } L^p_{L^q} \text{ for any } p,\, q \in [1,\infty) \\ \nabla_\Omega u_\epsilon &\rightharpoonup \nabla_\Omega u && \text{in } L^2_{L^2} \\ \mathcal{E}_\epsilon(u_\epsilon) &\rightharpoonup \chi && \text{in } L^2_{L^2} \end{aligned} \tag{3.20}$$

where only the first strong convergence listed requires an explanation. For that, we recall the following compactness result.

Theorem 3.3.7 (Theorem 5 in [114])**.** Let $B_0 \overset{c}{\hookrightarrow} B_1 \subset B_2$ be Banach spaces. If $p \in [1,\infty)$, $\{u_n\}$ is uniformly bounded in $L^1_{\text{loc}}(0,T;B_0)$ and

$$\|u_n(\cdot + h) - u_n(\cdot)\|_{L^p(0,T-h;B_2)} \to 0 \quad \text{as } h \to 0, \text{ uniformly in } n,$$

then u_n has a convergent subsequence in $L^p(0,T;B_1)$ (and in $C^0([0,T];B_1)$ if $p = \infty$).

Indeed, the point is to apply this theorem with $H^1(\Omega_0) \overset{c}{\hookrightarrow} L^1(\Omega_0) \subset L^1(\Omega_0)$, which gives us a subsequence $\tilde{u}_{\epsilon_j} \to \tilde{\rho}$ strongly in $L^1(0,T;L^1(\Omega_0))$. It follows that $u_{\epsilon_j} \to \rho$ in $L^1_{L^1}$, whence for a.a. t, $u_{\epsilon_{j_k}}(t) \to \rho(t)$ a.e. in $\Omega(t)$. We also know that for a.a. t, $|u_{\epsilon_{j_k}}(t)| \leq C$ a.e. in $\Omega(t)$ by Lemma 3.3.4, and so for a.a. t, the limit satisfies $|\rho(t)| \leq C$ a.e. in $\Omega(t)$ too. By Theorem 3.2.3, $u_{\epsilon_{j_k}} \to \rho$ in $L^p_{L^q}$ for all p, $q \in [1,\infty)$. Since $u_{\epsilon_{j_k}} \rightharpoonup u$ (subsequences have the same weak limit), it must be the case that $\rho = u$.

Let us now conclude the existence for bounded data.

Proof of Theorem 3.1.3. In ($\mathbf{P}_\epsilon$), we can test with a function $\eta \in W(H^1, L^2)$ with $\eta(T) = 0$, integrate by parts and then pass to the limit to obtain

$$-\int_0^T \int_{\Omega(t)} \dot{\eta}(t)\chi(t) + \int_0^T \int_{\Omega(t)} \nabla_\Omega u(t) \nabla_\Omega \eta(t) = \int_0^T \int_{\Omega(t)} f(t)\eta(t) + \int_{\Omega_0} e_0 \eta(0)$$

and it remains to be seen that $\chi \in \mathcal{E}(u)$ or equivalently $u = \mathcal{U}(\chi)$. By monotonicity of $\mathcal{E}_\epsilon$, we have for any $w \in L^2_{L^2}$

$$\int_0^T \int_{\Omega(t)} (\mathcal{E}_\epsilon(u_\epsilon) - w)(u_\epsilon - \mathcal{U}_\epsilon(w)) \geq 0.$$

Because $\mathcal{U}_\epsilon \to \mathcal{U}$ uniformly, for a.a. t, $\mathcal{U}_\epsilon(w(t)) \to \mathcal{U}(w(t))$ a.e. in $\Omega(t)$, and

$|\mathcal{U}_\epsilon(w)| \leq |w|$, and the dominated convergence theorem shows that $\mathcal{U}_\epsilon(w) \to \mathcal{U}(w)$ in $L^2_{L^2}$. Using this and (3.20), we can easily pass to the limit in this inequality and obtain

$$\int_0^T \int_{\Omega(t)} (\chi - w)(u - \mathcal{U}w) \geq 0 \quad \text{for all } w \in L^2_{L^2}.$$

By Minty's trick[1] [108, Lemma 2.13] we find $u = \mathcal{U}(\chi)$. To see why $\chi \in L^\infty_{L^\infty}$, we have from the estimate in Lemma 3.3.4 that for a.a. $t \in [0, T]$, $\|\mathcal{E}_\epsilon(\tilde{u}_\epsilon(t))\|_{L^\infty(\Omega_0)} \leq C$, giving $\mathcal{E}_\epsilon(\tilde{u}_\epsilon(t)) \overset{*}{\rightharpoonup} \tilde{\zeta}(t)$ in $L^\infty(\Omega(t))$ and, by weak-* lower semicontinuity, also $\|\tilde{\zeta}(t)\|_{L^\infty(\Omega(t))} \leq C$ for a.a. t, and we just need to identify $\tilde{\zeta} \in \mathcal{E}(\tilde{u})$. It follows from (3.20) that $\mathcal{E}_\epsilon(u_\epsilon) \to \chi$ in $L^2_{H^{-1}}$ by Aubin–Lions, and so for a.e. t and for a subsequence (not relabelled), $\mathcal{E}_\epsilon(u_\epsilon(t)) \to \chi(t)$ in $H^{-1}(\Omega(t))$. This allows us to conclude that $\chi = \zeta$ (the weak-* convergence of $\mathcal{E}_\epsilon(\tilde{u}_\epsilon(t))$ to $\tilde{\zeta}(t)$ also gives weak convergence in any $L^p(\Omega(t))$ to the same limit). $\square$

3.3.3 Continuous dependence and uniqueness of bounded weak solutions

The next lemma allows us to drop the requirement for our test functions to vanish at time T.

Lemma 3.3.8. If (u, e) is a bounded weak solution (satisfying (3.4)), then (u, e) also satisfies

$$\int_{\Omega(T)} e(T)\eta(T) - \int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) + \int_0^T \int_{\Omega(t)} \nabla_\Omega u(t) \nabla_\Omega \eta(t) = \int_0^T \int_{\Omega(t)} f(t)\eta(t) + \int_{\Omega_0} e_0 \eta(0)$$

for all $\eta \in W(H^1, L^2)$.

Proof. To see this, for $s \in (0, T]$, consider the function $\chi_{\epsilon,s}(t) = \min\left(1, \epsilon^{-1}(s-t)^+\right)$ (see Figure 3.3.1) which has a weak derivative $\chi'_{\epsilon,s}(t) = -\epsilon^{-1}\chi_{(s-\epsilon,s)}(t)$. Take the test function in (3.4) to be $\chi_{\epsilon,T}\eta$ where $\eta \in W(H^1, L^2)$, send $\epsilon \to 0$ and use the Lebesgue differentiation theorem. $\square$

We can finally prove Theorem 3.1.4, that of uniqueness and continuous dependence for bounded data.

[1]The trick, applied to our setting, is as follows. First, pick $w = \chi + \delta x$ for arbitrary $x \in L^2_{L^2}$ and $\delta > 0$, then the integrand simplifies to $\delta x(u - \mathcal{U}(\chi - \delta x))$. Divide by δ and then send $\delta \to 0$ using the continuity of $\mathcal{U}$ to find $(x, u - \mathcal{U}\chi)_{L^2_{L^2}} \geq 0$.

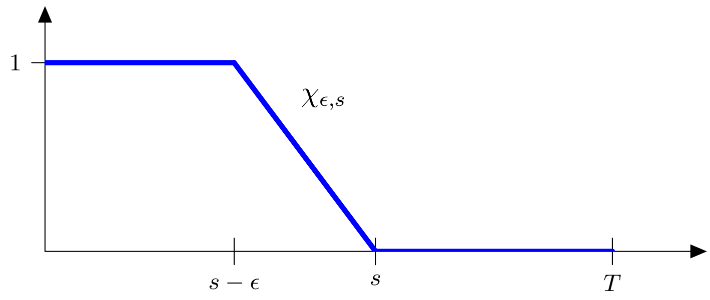

Figure 3.3.1: The function $\chi_{\epsilon,s}$

Proof of Theorem 3.1.4. We can prove the continuous dependence as in [81, Chapter V, §9]. As explained in Lemma 3.3.8, we drop the requirement $\eta(T) = 0$ in our test functions and we now suppose that $\Delta_\Omega \eta \in L^2_{L^2}$. Suppose for $i = 1, 2$ that (u_i, e_i) is the solution to the Stefan problem with data (f_i, u_0^i), so

$$\begin{aligned}\int_{\Omega(t)} (e_i(t) - e_2(t))\eta(t) - \int_0^t \int_{\Omega(\tau)} (\dot{\eta}(\tau)(e_1(\tau) - e_2(\tau)) + (u_1(\tau) - u_2(\tau))\Delta_\Omega \eta(\tau)) \\ = \int_0^t \int_{\Omega(\tau)} (f_1(\tau) - f_2(\tau))\eta(\tau) + \int_{\Omega_0} (e_0^1 - e_0^2)\eta(0). \qquad (3.21)\end{aligned}$$

Define $a = (u_1 - u_2)/(e_1 - e_2)$ when $e_1 \neq e_2$ and $a = 0$ otherwise, and note that $0 \leq a(x,t) \leq 1$. Let η_ϵ solve in $\cup_{\tau \in (0,t)} \{\tau\} \times \Omega(\tau)$ the equation

$$\begin{aligned}\partial^\bullet_\tau \eta_\epsilon(\tau) + (a_\epsilon(x,\tau) + \epsilon)\Delta_\Omega \eta_\epsilon(\tau) = 0 \\ \eta_\epsilon(t) = \xi \quad \text{on } \Omega_0\end{aligned} \qquad (3.22)$$

with $\xi \in C^1(\Omega_0)$ and where a_ϵ satisfies $\phi_{-(\cdot)} a_\epsilon \in C^2([0,T] \times \Omega_0)$ and $0 \leq a_\epsilon \leq 1$ a.e. and $\|a_\epsilon - a\|_{L^2(Q)} \leq \epsilon$. This is well posed by Lemma 3.2.6. Equation (3.21) can be written in terms of a_ϵ, and if we choose $\eta = \eta_\epsilon$ and use (3.22), we find

$$\begin{aligned}\int_{\Omega(t)} (e_1(t) - e_2(t))\xi \leq \|e_1 - e_2\|_{L^\infty_{L^\infty}} \int_0^t \int_{\Omega(\tau)} (|a(x,\tau) - a_\epsilon(x,\tau)| + \epsilon)|\Delta_\Omega \eta_\epsilon(\tau)| \\ + \|\xi\|_{L^\infty(\Omega_0)} \int_0^t \|f_1(\tau) - f_2(\tau)\|_{L^1(\Omega(\tau))} \\ + \|\xi\|_{L^\infty(\Omega_0)} \int_{\Omega_0} |e_0^1 - e_0^2| \qquad (3.23)\end{aligned}$$

using the L^∞ bound from Lemma 3.2.6. We can estimate the first integral on the right hand side:

$$\int_0^t \int_{\Omega(\tau)} |a(x,\tau) - a_\epsilon(x,\tau)||\Delta_\Omega \eta_\epsilon(\tau)|$$
$$\leq \sqrt{\epsilon}\, \|a - a_\epsilon\|_{L^2_{L^2}} \sqrt{(2+\epsilon)(1+e^{2C_{\mathbf{w}}(2+\epsilon)t})}\, \|\nabla_\Omega \xi\|_{L^2(\Omega_0)}$$

and

$$\int_0^t \int_{\Omega(\tau)} |\epsilon \Delta_\Omega \eta_\epsilon| \leq \sqrt{t|\Omega|\epsilon(2+\epsilon)(1+e^{2C_{\mathbf{w}}(2+\epsilon)t})} \left(\int_{\Omega_0} |\nabla_\Omega \xi|^2\right)^{\frac{1}{2}}$$

by the results in Lemma 3.2.6. Sending $\epsilon \to 0$ in (3.23) gives us (recalling $\xi \leq 1$),

$$\int_{\Omega(t)} (e_1(t) - e_2(t))\xi \leq \int_0^t \|f_1(\tau) - f_2(\tau)\|_{L^1(\Omega(\tau))} + \|e_0^1 - e_0^2\|_{L^1(\Omega_0)}.$$

Now pick $\xi = \xi_n$ where $\xi_n(x) \to \text{sign}(e_1(t,x) - e_2(t,x)) \in L^2(\Omega(t))$ a.e. in $\Omega(t)$. $\square$

3.3.4 Well-posedness of weak solutions

So far, we have proved well-posedness only for problems for which the initial data and the right hand side data are bounded. Now we prove the main result, Theorem 3.1.5, which we recall here.

Theorem 3.3.9 (Well-posedness of weak solutions)**.** If $f \in L^1_{L^1}$, $e_0 \in L^1(\Omega_0)$ and $|\Omega| := \sup_{s\in[0,T]} |\Omega(s)| < \infty$, then there exists a unique weak solution to (3.1). Furthermore, if for $i = 1, 2$, $(u^i, e^i) \in L^1_{L^1} \times L^1_{L^1}$ are two weak solutions of (3.1) with data $(f^i, e_0^i) \in L^1_{L^1} \times L^1(\Omega_0)$, then

$$\|e^1 - e^2\|_{L^1_{L^1}} \leq C_T \left(\|f^1 - f^2\|_{L^1_{L^1}} + \|e_0^1 - e_0^2\|_{L^1(\Omega_0)}\right).$$

Proof of Theorem 3.1.5. Suppose $(e_0, f) \in L^1(\Omega_0) \times L^1_{L^1}$ are data and consider functions $e_{0n} \in L^\infty(\Omega_0)$ and $f_n \in L^\infty_{L^\infty}$ satisfying

$$(f_n, e_{0n}) \to (f, e_0) \quad \text{in } L^1_{L^1} \times L^1(\Omega_0).$$

The existence of f_n holds because by density, there exist $\tilde{f}_n \in C^0([0,T] \times \Omega_0)$ such that $\tilde{f}_n \to \tilde{f}$ in $L^1((0,T) \times \Omega_0) \equiv L^1(0,T; L^1(\Omega_0))$. Denote by (u_n, e_n) the respective (bounded weak) solutions to the Stefan problem with the data (e_{0n}, f_n). By virtue of these solutions satisfying the continuous dependence result, it follows that $\{e_n\}_n$ is a Cauchy sequence in $L^1_{L^1}$ and thus $e_n \to \chi$ in $L^1_{L^1}$ for some χ. Recall that

$|u_n| = |\mathcal{U}(e_n)| \leq |e_n|$ so by consideration of an appropriate Nemytskii map, we find $u_n = \mathcal{U}(e_n) \to \mathcal{U}(\chi)$. Now we can pass to the limit in

$$-\int_0^T \int_{\Omega(t)} \dot{\eta}(t) e_n(t) - \int_0^T \int_{\Omega(t)} u_n(t) \Delta_\Omega \eta(t) = \int_0^T \int_{\Omega(t)} f_n(t)\eta(t) + \int_{\Omega_0} e_{n0}\eta(0)$$

and doing so gives

$$-\int_0^T \int_{\Omega(t)} \dot{\eta}(t)\chi(t) - \int_0^T \int_{\Omega(t)} \mathcal{U}(\chi(t)) \Delta_\Omega \eta(t) = \int_0^T \int_{\Omega(t)} f(t)\eta(t) + \int_{\Omega_0} e_0\eta(0)$$

and overall this shows that there exists a pair $(\chi, \mathcal{E}^{-1}(\chi)) \in L^1_{L^1} \times L^1_{L^1}$ which is a weak solution of the Stefan problem. For these integrals to make sense, we need $\eta \in W^1(L^\infty \cap H^2, L^\infty)$ with $\Delta_\Omega \eta \in L^\infty_{L^\infty}$.

Now suppose that (u^1, e^1) and (u^2, e^2) are two weak solutions of class L^1 to the Stefan problem with data (f^1, e^1_0) and (f^2, e^2_0) in $L^1_{L^1} \times L^1(\Omega_0)$ respectively. We know that there exist approximations (f^1_n, e^1_{0n}), $(f^2_n, e^2_{0n}) \in L^\infty_{L^\infty} \times L^\infty(\Omega_0)$ of the data satisfying

$$(f^1_n, e^1_{0n}) \to (f^1, e^1_0) \quad \text{and} \quad (f^2_n, e^2_{0n}) \to (f^2, e^2_0) \quad \text{in } L^1_{L^1} \times L^1(\Omega_0).$$

These approximate data give rise to the approximate solutions e^1_n and e^2_n both of which are elements of $L^\infty_{L^\infty}$. It follows from above that $e^1_n \to e^1$ and $e^2_n \to e^2$ in $L^1_{L^1}$. Now consider the continuous dependence result that e^1_n and e^2_n satisfy:

$$\|e^1_n - e^2_n\|_{L^1_{L^1}} \leq T \left(\|f^1_n - f^2_n\|_{L^1_{L^1}} + \|e^1_{0n} - e^2_{0n}\|_{L^1(\Omega_0)} \right). \tag{3.24}$$

Regarding the right hand side, by writing $e^1_{0n} - e^2_{0n} = e^1_{0n} - e^1_0 + e^1_0 - e^2_0 + e^2_0 - e^2_{0n}$, (and similarly for the f^i_n) and using triangle inequality, along with the fact that $e^1_n - e^2_n \to e^1 - e^2$ in $L^1_{L^1}$, we can take the limit in (3.24) as $n \to \infty$ and we are left with what we desired. $\square$

Chapter 4

A fractional porous medium equation on an evolving surface

4.1 Introduction

For each $t \in [0,T]$, let $\Gamma(t) \subset \mathbb{R}^{d+1}$ be a smooth and compact d-dimensional hypersurface without boundary evolving with a given velocity field $\mathbf{w}$. In this chapter, we are interested in the well-posedness of the fractional porous medium equation

$$\begin{aligned} \dot{u}(t) + (-\Delta_{\Gamma(t)})^{1/2}(u^m(t)) + u(t)\nabla_{\Gamma(t)} \cdot \mathbf{w}(t) = 0 \quad &\text{on } \Gamma(t) \\ u(0) = u_0 \quad &\text{on } \Gamma_0 := \Gamma(0) \end{aligned} \tag{4.1}$$

for $m \geq 1$, where u_0 is a given initial data, $u^m := |u|^{m-1}u$ as usual, and $(-\Delta_{\Gamma(t)})^{1/2}$ is the square root of the Laplace–Beltrami operator on $\Gamma(t)$, which is a nonlocal first order elliptic pseudodifferential operator [110, 113, 118].

If the fractional Laplacian in (4.1) is replaced with the ordinary Laplace–Beltrami operator $-\Delta_{\Gamma(t)}$, (4.1) would be a porous medium equation on an evolving surface. The porous medium equation can be used to model, amongst other applications [93], gas flow through porous media [82], groundwater infiltration [23], heat radiation, population dynamics, and plasma physics [121, §1–2]. It is perhaps the simplest example to write down of a heat equation with a nonlinear diffusion. On stationary domains, porous medium equations have, of course, attracted a considerable and well-developed literature. We refer the reader to the book [121] by Vázquez which is a comprehensive study of the mathematical analysis and key properties of the equation (and it also contains many references). Results on the porous medium equation on manifolds can be found in [121, §11.5] and [18]. We will also say a few words about the non-fractional moving case in the conclusion of this chapter.

The investigation of *fractional* porous medium equations was initiated in the first paper [44] on the topic where De Pablo, Quirós, Rodríguez and Vázquez examined such a problem on $\mathbb{R}^d$ involving the square root of the Laplacian and gave a complete theory of the equation, and indeed, our work is motivated by the results in that paper and we aim to give a similar analogous theory to the stationary case in [44]. Fractional diffusion models anomolous diffusion and the fractional porous medium equation appears in statistical mechanics and heat control (see [44, 45] for references). Understanding the effects of the nonlocal diffusion with arguably the prototypical nonlinearity in a rigorous mathematical sense is also a motivation for studying this problem (there are some interesting differences: for example, free boundaries do not arise in the fractional porous medium equation [44, §5.2] unlike the non-fractional case). With regards to our choice of posing the problem on an evolving surface: as we have explained before in previous chapters, physical models are often more realistically formulated on curved spaces which are changing, since this is precisely the situation in reality. The challenge of the analysis in the moving framework is also more involved which compounds the interest in the problem (4.1).

Remark 4.1.1 (The standard porous medium equation and the fractional heat equation)**.** The reader may wonder why we did not first study on an evolving surface either a non-fractional porous medium equation or a fractional heat equation. The fractional heat equation (i.e., the $m = 1$ case in (4.1)) by its linearity means that we can simply apply the theory of Chapter 1 once the functional framework in §4.2–4.4 has been set up. This should result in a solution with a weak time derivative in a dual space. To derive the regularity result given by Theorem 1.4.8, we would need to prove a transport formula for the fractional Laplacian, i.e., an appropriate expression for

$$\frac{d}{dt}\int_{\Gamma(t)} |(-\Delta_{\Gamma(t)})^{1/2}u|^2$$

analogous to the Dirichlet inner product case proved in [48] (see (2.15)). This is an open issue and its resolution would be useful. The non-fractional porous medium equation can be handled in essentially the same way as the Stefan problem in Chapter 3. The porous medium case is more difficult, however, since the power nonlineary is not globally Lipschitz and more work is needed in order to obtain uniform bounds. The arguments we give in this chapter can be adapted to the non-fractional setting; see the concluding remarks for more details.

In [44], the existence was proved by discretisation in time of a localised formulation of the equation and then the application of the Crandall–Liggett theorem[1]

[1]The theorem by Crandall and Liggett gives conditions on the *nonlinear* elliptic operator under

[39]. Those results were generalised in [45] to a wider range of fractional powers of the Laplacian $(-\Delta)^s$ with exponent $s \in (0,1)$ on a stationary domain $\Omega \subseteq \mathbb{R}^d$ using the extension method introduced by Caffarelli and Silvestre in [29]. Existence was proved in [19] (for a more general nonlinearity) in a different way through the theory of semigroups and maximal monotone operators. Our model (4.1) differs from all of the aforementioned works since it is on a *moving* space.

Before moving on, let us quickly discuss some other related works. Recently, Bonforte and Vázquez have considered in [20] equations of the form

$$\dot{u} + \mathcal{L}F(u) = 0$$

on bounded domains, where $\mathcal{L}$ is in a class of general linear operators that includes fractional Laplacians and F is a nonlinear mapping that includes the case of the porous medium nonlinearity. The authors propose a different type of weak solution (generalising a notion they defined earlier in [19]) and prove well-posedness and other properties. There is also work on variants of nonlocal porous medium equations such as those with variable density [101, 102] and different fractional operators [13]. We also mention [6, 30, 117, 92] where elliptic fractional problems are studied in the setting of the Laplacian on a bounded domain with Neumann boundary conditions, and [75] where a degenerate parabolic equation arising in crack dynamics is considered, again in the Neumann setting. One can also find numerical and finite element analysis for elliptic and parabolic problems in [96, 95]. As is evident, there has been an extraordinary amount of activity in fractional diffusion problems in the last decade or so. A good survey of recent and current output involving nonlinear fractional diffusion can be found in the articles [122, 123].

In terms of the analysis, a common preliminary step when working with half-Laplacians is to rewrite the problem locally using a Dirichlet-to-Neumann map [28, 8, 115, 34]. We will also reformulate (4.1) using such a map; this step is likewise performed in [44, 45] but from here on, the type of approaches taken in [44, 45] are problematic in our setting because of the additional complexity engendered by the evolving domain. For example, one could attempt to pull back the problem onto a reference domain (the resulting expression is not too cumbersome if the evolution of $\Gamma(t)$ is prescribed particularly agreeably), discretise the equation and try to employ an appropriate time-dependent version of Crandall–Liggett [40, 62, 83] to the resulting equation (which will have time-dependent coefficients) but these theorems are difficult to apply even when the evolution of the domains is highly

which one can pass to the limit in the discretised equation, resulting in a mild solution. It is a nonlinear generalisation of the well-known linear case (see [72]).

simplified. Therefore, we choose a different way to approach this problem, which we shall outline below, starting from the foundations. To our knowledge, the type of approach developed in this chapter has not been used before in the fractional setting, even in the stationary case. The challenges and peculiarities that arise due to the moving domain will be highlighted in due course.

Before we proceed, let us remark that fractional Laplace–Beltrami operators on various classes of manifolds have been studied in [8, 115, 34] through extension problems in the style of Caffarelli–Silvestre [29], but a convenient work detailing all the relevant properties of the half-Laplacian on closed manifolds in a Sobolev space setting appears lacking, so this work is useful also in this respect. With this in mind, it is worth emphasising that the first part of this chapter, comprising of §4.2–4.4, is independent of the second part which consists of §4.5 and §4.6, and indeed the reader can read the first part in isolation. The first part can be of use for other fractional diffusion problems on (evolving) manifolds and the second part can be thought of as an application of the first part. See the outline below for more details.

Novelty of the work Let us briefly state what is new in this chapter and what has already been considered by others. The results on the harmonic extension on manifolds in §4.2 are what one would expect, however, as mentioned above, it is remarkably difficult to find a text where all the various functional analysis is done (there does not appear to be much work in the literature on fractional diffusion on manifolds). The results on the truncated harmonic extension problem are new, as are the results on the harmonic extensions on evolving surfaces in §4.4 (the method we used there to obtain time measurability is one we have not seen elsewhere). Some of the functional analysis in §4.3 is also new. The techniques used to pass to the limit in the latter parts of this chapter are relatively standard but their application requires some technical details (such as carefully picking the correct test functions) amongst which some do not arise in the stationary setting, and some of it is delicate.

4.1.1 Reformulation of the equation and main results

A natural way to define $(-\Delta_{\Gamma(t)})^{1/2}$ is through a spectral definition which we describe now in greater generality. Indeed, suppose that

$$(M, g) \text{ is a connected closed smooth Riemannian manifold} \qquad (A_M)$$

and let $(\varphi_k, \lambda_k)_{k\in\mathbb{N}}$ be the normalised eigenpairs of the Laplacian $-\Delta_M$ so that $-\Delta_M\varphi_k = \lambda_k\varphi_k$ for each k; it follows that $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ... \nearrow \infty$ and

$\varphi_0 \equiv |M|^{-1/2}$ [76, Theorem 3.2.1]. The φ_k form an orthonormal basis of $L^2(M)$ and are orthogonal in $H^1(M)$. For smooth functions u, define

$$(-\Delta_M)^{1/2}u := \sum_{k=1}^{\infty} \lambda_k^{1/2}(u, \varphi_k)_{L^2(M)}\varphi_k. \tag{4.2}$$

The operator $(-\Delta_M)^{1/2}$ can be defined in a weaker sense through the action

$$\langle(-\Delta_M)^{1/2}u, v\rangle := \sum_{k=1}^{\infty} \lambda_k^{1/2}(u, \varphi_k)_{L^2(M)}(v, \varphi_k)_{L^2(M)} \tag{4.3}$$

which is sensible whenever u and v belong to the Hilbert space

$$H(M) := \left\{ u \in L^2(M) \mid \sum_{k=1}^{\infty} \lambda_k^{1/2}|(u, \varphi_k)_{L^2(M)}|^2 < \infty \right\} \tag{4.4}$$

endowed with the inner product

$$(u, v)_{H(M)} := (u, v)_{L^2(M)} + \sum_{k=1}^{\infty} \lambda_k^{1/2}(u, \varphi_k)_{L^2(M)}(v, \varphi_k)_{L^2(M)}.$$

It is useful to have a Sobolev characterisation of the space $H(M)$; in Lemma 4.2.9, we will see that

$$H(M) = H^{1/2}(M) = B_{22}^{1/2}(M) = (L^2(M), W^{1,2}(M))_{1/2},$$

i.e., $H(M)$ is exactly the fractional Sobolev space $H^{1/2}(M)$ (see [120, §7.2.2, §7.3.1, §7.4.5] for more details on the second and third equalities). In the later sections, we will be working on hypersurfaces so it is convenient for our purposes to introduce the Sobolev–Slobodeckiĭ space $W^{1/2,2}(\Gamma)$ (where Γ is a sufficiently smooth hypersurface) defined using the Gagliardo norm (see §2 and references therein):

$$\begin{aligned} W^{1/2,2}(\Gamma) := \Bigg\{ u \in L^2(\Gamma) \mid \|u\|^2_{W^{1/2,2}(\Gamma)} &:= \int_\Gamma |u(x)|^2 \, \mathrm{d}\sigma(x) \\ &+ \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^2}{|x - y|^n} \, \mathrm{d}\sigma(x)\mathrm{d}\sigma(y) < \infty \Bigg\}. \end{aligned}$$

Of course, this space is equivalent to $H^{1/2}(\Gamma)$ with an equivalence of norms (see [127, §I.4.2 and Theorem 5.2 of §I.5.1], [86, Theorem 7.7, Chapter 1], [86, Chapter 1, §15] and [71, §1.3.3]), but it is important to distinguish between these spaces when $\Gamma = \Gamma(t)$ is time-dependent because the constants in the equivalence of norms

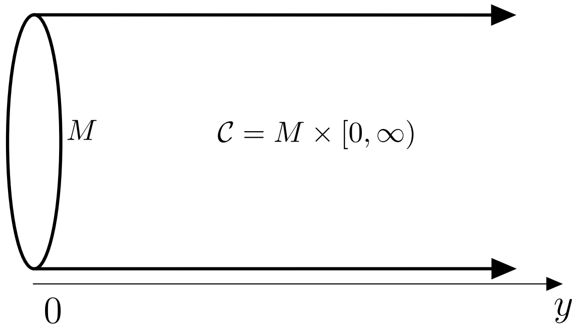

Figure 4.1.1: The semi-infinite cylinder $\mathcal{C}$

will depend on t in an unknown way.

The spectral definition of $(-\Delta_M)^{1/2}$ in (4.2) is not particularly amenable to a convenient theory of weak solutions; however, there is a way to localise the fractional Laplacian (see [8, 115, 34]). With $\mathcal{C} := M \times [0,\infty)$ (see Figure 4.1.1) and $\bar{g}$ denoting the trivial product metric on $\mathcal{C}$, consider the problem

$$\Delta_{\bar{g}} v = 0 \quad \text{on } \mathcal{C}, \qquad v|_{\partial\mathcal{C}} = u \tag{4.5}$$

where $\partial\mathcal{C} = M \times \{0\}$. Whenever u belongs to $H(M)$, the equation has a unique weak solution $v = \overline{\mathcal{E}}u$, called the harmonic extension of u.

This harmonic extension $\overline{\mathcal{E}}u$ belongs in general not to $H^1(\mathcal{C})$ but to the larger space

$$X(\mathcal{C}) := \overline{H^1(\mathcal{C})}^{\|\cdot\|_{X(\mathcal{C})}} \text{ where } \|v\|^2_{X(\mathcal{C})} := \|\nabla_{\bar{g}} v\|^2_{L^2(\mathcal{C})} + \|\mathcal{T}v\|^2_{L^2(M)} \quad \text{for } v \in H^1(\mathcal{C}) \tag{4.6}$$

with $\mathcal{T}\colon H^1(\mathcal{C}) \to H(M)$ denoting the trace map onto $M \times \{0\}$, so that $\overline{\mathcal{E}}\colon H(M) \to X(\mathcal{C})$ (this type of space $X(\mathcal{C})$ was first defined in a different setting by Stinga and Volzone in [117]). As we shall see in Lemma 4.2.6, the fractional Laplacian is recovered as a Dirichlet-to-Neumann map:

$$\langle (-\Delta_M)^{1/2} u, v \rangle_{H(M)^*,H(M)} = \left\langle \frac{\partial(\overline{\mathcal{E}}u)}{\partial\nu}\Big|_{y=0}, v \right\rangle_{H(M)^*,H(M)},$$

where $\nu = (0,-1)$ is the outward normal to $\mathcal{C}$. All of this will be laid out in detail in §4.2.

Setting $\Psi(r) := |r|^{m-1}r$ and $\mathcal{C}(t) := \Gamma(t) \times [0,\infty)$, the above characterisation implies that one can rewrite (4.1) as

$$\begin{aligned}
\dot{u}(t) + u(t)\nabla_{\Gamma(t)} \cdot \mathbf{w}(t) + \frac{\partial v(t)}{\partial \nu(t)} &= 0 && \text{on } \partial\mathcal{C}(t) \\
v(t) &= \overline{\mathcal{E}}_t(\Psi(u(t))) && \\
u(0) &= u_0 && \text{on } \Gamma_0
\end{aligned} \tag{\textbf{P}}$$

where $\overline{\mathcal{E}}_t$ is the map $\overline{\mathcal{E}}$ with the manifold M chosen to be $\Gamma(t)$ and $\nu(t) = (0, -1)$ is outward normal to $\mathcal{C}(t)$. Regarding the regularity of $\{\Gamma(t)\}_{t\in[0,T]}$, we will assume Assumption 4.3.1 on p. 137 and that

$$\text{there exists a constant } \lambda_1 > 0 \text{ such that } \lambda_1(t) \geq \lambda_1 \text{ for all } t \in [0, T] \tag{A_λ}$$

where $\lambda_k(t)$ denotes the k-th eigenvalue of $-\Delta_{\Gamma(t)}$; see Remark 4.3.2.

In §4.3 we shall make clear the assumptions on the evolution of the hypersurface $\Gamma(t)$ and we shall check that the following spaces are well defined.

Space L^p_Y	Formed from $\{Y(t)\}_{t\in[0,T]}$
$L^p_{L^q}$	$\{L^q(\Gamma(t))\}_{t\in[0,T]}$
$L^2_{W^{1/2,2}}$	$\{W^{1/2,2}(\Gamma(t))\}_{t\in[0,T]}$
$L^2_{L^2(\mathcal{C})}$	$\{L^2(\mathcal{C}(t))\}_{t\in[0,T]}$
$L^2_{H^1(\mathcal{C})}$	$\{H^1(\mathcal{C}(t))\}_{t\in[0,T]}$
$L^2_{X(\mathcal{C})}$	$\{X(\mathcal{C}(t))\}_{t\in[0,T]}$

The space $\mathbb{W}(Y, Z) := \{u \in L^2_Y \mid \dot{u} \in L^2_Z\}$ with $\dot{u}$ the weak time or material derivative refers to the evolving Sobolev–Bochner space for which we had previously used the notation $W(Y, Z)$.

In order to obtain measurability in time of $t \mapsto \overline{\mathcal{E}_t}(\Psi(u(t)))$ for $u \in L^2_{W^{1/2,2}}$ (recall that each $\overline{\mathcal{E}}_t$ was defined individually at each moment in time as the harmonic extension on $\Gamma(t)$), we will consider in §4.4 the "$L^2_{X(\mathcal{C})}$ harmonic extension" problem: given $u \in L^2_{W^{1/2,2}}$, find $\overline{\mathbb{E}}u = v \in L^2_{X(\mathcal{C})}$ such that

$$\Delta_{\bar{g}} v = 0, \quad \overline{\mathbb{T}}v = u \tag{4.7}$$

holds with $\overline{\mathbb{T}}\colon L^2_{X(\mathcal{C})} \to L^2_{W^{1/2,2}}$ the trace map. Then we will show that $(\overline{\mathbb{E}}u)(t) = \overline{\mathcal{E}}_t u(t)$ for almost all t, which gives the desired measurability. Of course, in the stationary setting, this issue of measurability would not arise and there would be no need to consider (4.7). Now we can think about what we mean by a weak solution. In what follows, given $\eta \in L^2_{W^{1/2,2}}$, we denote by $E\eta \in L^2_{H^1(\mathcal{C})}$ an arbitrary extension of η that satisfies $\overline{\mathbb{T}}E\eta = \eta$.

Definition 4.1.2 (Weak solution)**.** A weak solution of (**P**) is a function $u \in L^\infty_{L^\infty}$ with $\overline{\mathbb{E}}(\Psi(u)) \in L^2_{X(\mathcal{C})}$ satisfying

$$-\int_0^T \int_{\Gamma(t)} \dot{\eta}(t)u(t) + \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi(u(t)))\nabla_{\bar{g}(t)}(E\eta)(t) = \int_{\Gamma_0} u_0\eta(0)$$

for all $\eta \in \mathbb{W}(W^{1/2,2}, L^2)$ with $\eta(T) = 0$.

Remark 4.1.3 (Formal derivation of weak formulation)**.** Formally, to obtain (with the aid of a generalised Green's formula) a weak form of (4.1), we have two options. (1) One can multiply the first equation in (**P**) by a test function $\eta(t) \in H^{1/2}(\Gamma(t))$ and integrate by parts to obtain

$$\begin{aligned}
\langle \dot{u}(t), \eta(t)\rangle + \int_{\Gamma(t)} u(t)\eta(t)\nabla_\Gamma \cdot \mathbf{w} &= -\left\langle \frac{\partial v(t)}{\partial \nu(t)}, \eta(t)\right\rangle \\
&= -\int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} v(t)\nabla_{\bar{g}(t)}\hat{\eta}(t) \\
&= -\int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi(u(t)))\nabla_{\bar{g}(t)}\hat{\eta}(t)
\end{aligned}$$

where $\hat{\eta}(t) \in H^1(\mathcal{C}(t))$ is an extension of $\eta(t)$ to the cylinder $\mathcal{C}(t)$.
(2) Or one could test the weak formulation satisfied by $\overline{\mathcal{E}}_t(\Psi(u(t)))$ by a test function $\zeta(t) \in H^1(\mathcal{C}(t))$ and use the first equation of (**P**):

$$\begin{aligned}
0 &= \langle \Delta_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi(u(t))), \zeta(t)\rangle \\
&= -\int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi(u(t)))\nabla_{\bar{g}(t)}\zeta(t) + \int_{\Gamma(t)} \frac{\partial \overline{\mathcal{E}}_t(\Psi(u(t)))|_{y=0}}{\partial \nu(t)}\zeta(t)|_{y=0} \\
&= -\int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi(u(t)))\nabla_{\bar{g}(t)}\zeta(t) - \int_{\Gamma(t)} (\dot{u}(t) + u(t)\nabla_\Gamma \cdot \mathbf{w})\zeta(t)|_{y=0}.
\end{aligned}$$

Our definition of a weak solution is slightly weaker than the ones derived above. Let us mention the first viewpoint is more convenient for us since the weak formulation is defined in terms of boundary quantities.

We will prove the following theorem in §4.6, which is the main result of our work.

Theorem 4.1.4 (Well-posedness of the fractional porous medium equation)**.** Under Assumption 4.3.1 and (A_λ), given $u_0 \in L^\infty(\Gamma_0)$, there exists a unique weak solution

$u \in L^\infty_{L^\infty} \cap L^2_{W^{-1/2,2}}$ to $(\mathbf{P})$ with $\overline{\mathbb{E}}(\Psi(u)) \in L^2_{X(\mathcal{C})}$ (in the sense of Definition 4.1.2). Furthermore, we have the following properties:

1. Boundedness: for all $t \in [0,T]$, $u(t) \in L^\infty(\Gamma(t))$.

2. Conservation of mass: for all $t \in [0,T]$,

$$\int_{\Gamma(t)} u(t) = \int_{\Gamma_0} u_0.$$

3. L^1-contraction principle: if u_{01} and u_{02} are two pairs of initial data in $L^\infty(\Gamma_0)$, then the respective solutions u_1 and u_2 satisfy

$$\int_{\Gamma(t)} (u_1(t) - u_2(t))^+ \leq \int_{\Gamma_0} (u_{01} - u_{02})^+ \quad \text{for all } t \in [0,T].$$

An immediate consequence of the contraction principle is the following.

Corollary 4.1.5 (L^1-continuous dependence and comparison principle)**.** If u_{01} and u_{02} are two pairs of initial data in $L^\infty(\Gamma_0)$, then the respective weak solutions u_1 and u_2 of Theorem 4.1.4 satisfy the L^1-continuous dependence result

$$\int_{\Gamma(t)} |u_1(t) - u_2(t)| \leq \int_{\Gamma_0} |u_{01} - u_{02}| \quad \text{for all } t \in [0,T].$$

If $u_{01} \leq u_{02}$ a.e., then $u_1(t) \leq u_2(t)$ a.e. in $\Gamma(t)$ for all t.

Let us discuss how these results compare to those in the stationary case considered in [44, 45]. Theorem 4.1.4 and its corollary correspond to parts i, ii, iv and v of Theorem 2.2 of [44] and to Theorem 7.2 of [45] in the half-Laplacian setting. In short, these are the results that hold in the stationary case after changing the evolving function spaces to the standard ones. In terms of the proof, our methods are quite different, as already discussed earlier. Let us sketch the proof now.

4.1.2 Plan of the proof

In order to solve $(\mathbf{P})$ and prove Theorem 4.1.4, we will first approximate the nonlinearity Ψ by well-behaved smooth approximations Ψ_k and seek to solve $(\mathbf{P})$ with

Ψ replaced by Ψ_k. This directs us to study the non-degenerate problem

$$\begin{aligned} \dot{u}_\beta(t) + u_\beta(t)\nabla_{\Gamma(t)} \cdot \mathbf{w}(t) + \frac{\partial v_\beta(t)}{\partial \nu(t)} &= 0 && \text{on } \partial\mathcal{C}(t) \\ v_\beta(t) &= \overline{\mathcal{E}}_t(\beta(u_\beta(t))) \\ u_\beta(0) &= u_0 && \text{on } \Gamma_0 \end{aligned} \tag{$\mathbf{P}_\beta$}$$

where $\beta\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{aligned} &\beta(0) = 0, \beta \text{ is } C^2(\mathbb{R}) \text{ (and Lipschitz)} \\ &\beta', (\beta^{-1})', (\beta^{-1})'' \in L^\infty(\mathbb{R}), \text{ and} \\ &\text{there exist constants } C_{\beta'}, C_{\beta'_{inv}} > 0 \text{ with } \beta' \geq C_{\beta'} \text{ and } (\beta^{-1})' \geq C_{\beta'_{inv}}. \end{aligned} \tag{A_β}$$

To show well-posedness of $(\mathbf{P}_\beta)$ one could try a Galerkin method but a complication involving the unbounded cylinder $\mathcal{C}(t)$ arises due to the surface evolution, see Remark 4.5.5; this suggests truncating the cylinder $\mathcal{C}(t)$ in the unbounded direction. So we consider in §4.2.4 a truncated harmonic extension problem and show that its solution approximates the (untruncated) harmonic extension in some sense: given $u \in H(M)$, with $\overline{\mathcal{E}}_R u = v_R$ denoting the weak solution of

$$\Delta_{\bar{g}} v_R = 0 \quad \text{on } \mathcal{C}_R := M \times [0, R], \qquad v_R|_{M\times\{0\}} = u, \qquad v_R|_{M\times\{R\}} = 0, \tag{4.8}$$

we will show in §4.2.5 that $\nabla_{\bar{g}}\overline{\mathcal{E}}_R u \to \nabla_{\bar{g}}\overline{\mathcal{E}} u$ in $L^2(\mathcal{C})$ as $R \to \infty$. As with $\mathcal{E}_t$, we define $\overline{\mathcal{E}}_{R,t}$ as $\overline{\mathcal{E}}_R$ with $M = \Gamma(t)$ and $\mathcal{C}_R(t) := \Gamma(t) \times [0, R]$, and consider the following problem as an approximation of $(\mathbf{P}_\beta)$:

$$\begin{aligned} \dot{u}_{\beta R}(t) + u_{\beta R}(t)\nabla_{\Gamma(t)} \cdot \mathbf{w}(t) + \frac{\partial v_{\beta R}(t)}{\partial \nu(t)} &= 0 && \text{on } \Gamma(t) \times \{0\} \\ v_{\beta R}(t) &= \overline{\mathcal{E}}_{R,t}(\beta(u_{\beta R}(t))) \\ u_{\beta R}(0) &= u_0 && \text{on } \Gamma_0. \end{aligned} \tag{$\mathbf{P}_{\beta R}$}$$

We can define the spaces $L^2_{L^2(\mathcal{C}_R)}$ and $L^2_{H^1(\mathcal{C}_R)}$ on the truncated cylinder just like before, and consideration of an "$L^2_{H^1(\mathcal{C}_R)}$ truncated harmonic extension" problem like (4.7) in §4.4 will lead to a map $\overline{\mathbb{E}}_R$ and show the measurability in time of $\overline{\mathcal{E}}_{R,t}$. We will use the Galerkin method to solve $(\mathbf{P}_{\beta R})$ in §4.5.1, see Remark 4.5.2 where we explain the choice of our Galerkin approximation; this requires emphasis due to a technical difficulty in the evolution-dependent projection operators associated to the Galerkin basis. Then we will pass to the limit in R in §4.5.2 in order to settle $(\mathbf{P}_\beta)$ and the following theorem will be proved.

Theorem 4.1.6. Under Assumption 4.3.1, (A_λ), and (A_β), given $u_0 \in L^\infty(\Gamma_0)$, there exists a unique solution $u_\beta \in \mathbb{W}(W^{1/2,2}, W^{-1/2,2})$ to $(\mathbf{P}_\beta)$ with $u_\beta(0) = u_0$ and $\overline{\mathbb{E}}(\beta(u_\beta)) \in L^2_{X(\mathcal{C})}$ satisfying

$$\int_0^T \langle \dot{u}_\beta(t), \eta(t) \rangle + \int_0^T \int_{\Gamma(t)} u_\beta(t)\eta(t)\nabla_{\Gamma(t)} \cdot \mathbf{w}(t) \\ + \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} \overline{\mathcal{E}}_t(\beta(u_\beta(t)))\nabla_{\bar{g}(t)}(E\eta)(t) = 0 \tag{4.9}$$

for all $\eta \in L^2_{W^{1/2,2}}$, where the duality pairing is between the spaces $W^{-1/2,2}(\Gamma(t))$ and $W^{1/2,2}(\Gamma(t))$. Furthermore, mass is conserved and the L^1-contraction principle holds for almost all $t \in [0, T]$.

With β chosen to be the regularisation Ψ_k, this theorem gives us a sequence $\{u_k\}_{k\in\mathbb{N}}$ where $u_k \in \mathbb{W}(W^{1/2,2}, W^{-1/2,2})$ satisfies $u_k(0) = u_0$, $\overline{\mathbb{E}}(\Psi_k(u_k)) \in L^2_{X(\mathcal{C})}$, and the equation (4.9) with β replaced by Ψ_k and u_β replaced by u_k. Then we pass to the limit in k using energy estimates and the identification of limits is handled with the theory of subdifferentials of convex functionals in §4.6 where the proof of Theorem 4.1.4 is concluded.

In [44, 45], the authors prove results for existence with integrable data too, as well as other properties besides, including regularity, smoothing effects and extinction of solutions. As the next step to our results, studying regularity in time would be natural (and useful) but it appears difficult in our setting. We comment on this in more detail in the conclusion.

4.1.3 Outline

It is clear that we need to properly study the harmonic extension maps $\overline{\mathcal{E}}_t$ and $\overline{\mathcal{E}}_{R,t}$, which we take care of in §4.2 in the general setting of closed Riemannian manifolds. In §4.3 we shall check that the spaces L^p_Y listed above are well-defined and prove some preliminary functional analytic results. We then study the maps $\overline{\mathbb{E}}$ and $\overline{\mathbb{E}}_R$ in §4.4. After this, we tackle the non-degenerate problem $(\mathbf{P}_\beta)$ in §4.5 and then prove the main theorem in §4.6. We will finish with some concluding remarks in §4.7. Let us emphasise that §4.2 is useful more generally for fractional problems on closed manifolds and §4.3–4.5 are useful for fractional diffusion problems on (evolving) hypersurfaces. Only in §4.6 do we specialise to the porous medium equation.

4.1.4 Notation

We use the overline $\bar{\cdot}$ in different contexts. When applied to functions u, it means the spatial mean value: typically $\overline{u} = \frac{1}{|M|}\int_M u$ or $\overline{u} = \frac{1}{|\Gamma(t)|}\int_{\Gamma(t)} u$. When applied to symbols like $\mathbb{E}$ or $\mathcal{E}$ like $\overline{\mathbb{E}}$ or $\overline{\mathcal{E}}$, the meaning usually is that the map with the overline is a linear extension, for example, $\overline{\mathcal{E}}$ is a linear extension of $\mathcal{E}$ to a larger space. Symbols of the blackboard bold style like $\mathbb{E}$ refer to maps between the evolving Bochner spaces L^2_Y, whilst symbols of the calliographic style like $\mathcal{E}$ refer to maps between Sobolev spaces of the form $H^s(M)$. The notation $|\cdot|$ denotes a seminorm; usually the L^2 part of the corresponding norm is omitted.

As a convenience for the reader, we give here a list of the major notations and symbols that we use in this chapter along with the page number of definition or first usage.

Notation	Page
(A_M)	p. 107
$H(M)$	p. 108
$X(\mathcal{C})$	p. 109
(A_λ)	p. 110
$\mathcal{E}$, $\overline{\mathcal{E}}$	p. 118
$\mathcal{E}_R$, $\overline{\mathcal{E}}_R$	p. 128
$\mathscr{Z}_R$	p. 134

Notation	Page
$\mathcal{E}_t$, $\mathcal{E}_{R,t}$, $\overline{\mathcal{E}}_t$, $\overline{\mathcal{E}}_{R,t}$	p. 139
$\mathcal{T}_t$, $\mathcal{T}_{R,t,y=0}$, $\mathcal{T}_{R,t,y=R}$, $\overline{\mathcal{T}}_t$	p. 139
$\mathbb{T}$, $\overline{\mathbb{T}}$	p. 142
$\mathbb{T}_{R,y=0}$, $\mathbb{T}_{R,y=R}$	p. 143
$\mathbb{E}$, $\overline{\mathbb{E}}$	p. 149
$\mathbb{E}_R$, $\overline{\mathbb{E}}_R$	p. 152
$\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle_0$	p.155

4.2 The fractional Laplacian on compact Riemannian manifolds

The main purpose of this section is to realise the fractional Laplacian on a closed Riemannian manifold as the Dirichlet-to-Neumann map of a harmonic extension problem. Throughout this section, we assume that (M, g) is a Riemannian manifold as given in (A_M). We will define an operator $\overline{\mathcal{E}}\colon H^{1/2}(M) \to H^1(\mathcal{C})$ for this purpose. We also study a truncated harmonic extension by means of an operator $\overline{\mathcal{E}}_R\colon H^{1/2}(M) \to H^1(\mathcal{C}_R)$, and then prove that in some sense, $\mathcal{E}_R$ approximates $\mathcal{E}$. First, we begin with a brief discussion of Sobolev spaces on (semi-infinite) cylinders.

4.2.1 Sobolev spaces on semi-infinite cylinders

We can use the space $H^1(\mathcal{C})$ defined in [7] as the linear subspace of $L^1_{\text{loc}}(\mathcal{C})$ consisting of all v such that v and $\nabla_{\bar{g}} v$ belong to $L^2(\mathcal{C})$, and it is endowed with the natural norm. Equivalently, it can be defined as the linear subspace of $L^2(0,\infty;H^1(M))$ consisting of all v such that $v_y \in L^2(0,\infty;L^2(M))$. This is precisely the type of Sobolev–Bochner space whose theory was developed by Lions and Magenes [86, Chapter 1, §2.2]. There is a bounded linear surjective trace operator $\mathcal{T}\colon H^1(\mathcal{C}) \to H^{1/2}(M)$ [7, Theorem 18.1], [86, Theorem 3.2, Chapter 1], possessing a continuous right inverse. Similarly, the spaces $H^1(\mathcal{C}_R)$ can be defined on the truncated cylinder $\mathcal{C}_R = M \times [0,R]$. Theorem 3.1 of [86, Chapter 1] gives that $H^1(\mathcal{C}_R) \hookrightarrow C^0([0,R];H^{1/2}(M))$, so that the linear trace operators $\mathcal{T}_{R,y=0}$, $\mathcal{T}_{R,y=R}\colon H^1(\mathcal{C}_R) \to H^{1/2}(M)$ defined by $(\mathcal{T}_{R,y=0}v)(\cdot) := v(\cdot,0)$ and $(\mathcal{T}_{R,y=R}v)(\cdot) := v(\cdot,R)$ are also bounded. Furthermore $\mathcal{T}_{R,y=0}$ is surjective [86, Theorem 3.2, Chapter 1].

Lemma 4.2.1. If $v \in H^1(\mathcal{C})$, then $y \mapsto \overline{v}(y) = \frac{1}{|M|}\int_M v(y)$ is an element of $H^1(0,\infty)$ and thus $\overline{v} \in C^0([0,\infty))$.

Proof. A calculation verifies that $\overline{v} \in H^1(0,\infty)$, and Theorem 8.2 in [27] proves that each function in $H^1(0,\infty)$ has a unique continuous representative in $C^0([0,\infty))$. □

4.2.2 Fractional Sobolev spaces and the fractional Laplacian

The setting of a closed manifold is similar to the setting of Neumann boundary conditions on a bounded domain (see [117, 92]). As mentioned in the introduction, let (λ_k,φ_k) be the normalised eigenelements of the Laplace–Beltrami operator $-\Delta_M$. Since $(\varphi_k,\varphi_0)_{L^2(M)} = 0$ for $k \neq 0$, each φ_k for $k \neq 0$ has mean value zero. We also have $\|\varphi_k\|^2_{H^1(M)} = 1 + \lambda_k$ which implies that

$$H^1(M) = \left\{ u \in L^2(M) \mid \|u\|^2_{H^1(M)} = \sum_{k=0}^{\infty} (1+\lambda_k)|(u,\varphi_k)_{L^2(M)}|^2 < \infty \right\},$$

and it is clear that for $u \in H^1(M)$,

$$-\Delta_M u = \sum_{k=1}^{\infty} \lambda_k (u,\varphi_k)_{L^2(M)} \varphi_k \quad \text{in } H^{-1}(M)$$

with

$$\langle -\Delta_M u, v\rangle_{H^{-1}(M),H^1(M)} = \sum_{k=1}^{\infty} \lambda_k (u,\varphi_k)_{L^2(M)} (v,\varphi_k)_{L^2(M)}. \tag{4.10}$$

With the Hilbert space $H(M)$ as in (4.4), the last two identities inspire us to define $(-\Delta_M)^{1/2}\colon H(M) \to H(M)^*$ by (4.2) with the action (4.3). For u, $v \in H(M)$, it is easy to see the integration by parts formula

$$\langle(-\Delta_M)^{1/2}u, v\rangle_{H(M)^*,H(M)} = \int_M (-\Delta_M)^{1/4}u(-\Delta_M)^{1/4}v$$

where $\langle(-\Delta_M)^{1/4}u, v\rangle := \sum_{k=1}^{\infty} \lambda_k^{1/4}(u,\varphi_k)_{L^2(M)}(v,\varphi_k)_{L^2(M)}$, and we have

$$\left\|(-\Delta_M)^{1/4}u\right\|_{L^2}^2 = \sum_{k=1}^{\infty} \lambda_k^{1/2}|(u,\varphi_k)_{L^2(M)}|^2 = |u|_{H(M)}^2.$$

4.2.3 The harmonic extension problem

Recall the problem (4.5): given $u \in H(M)$, we want to find $v\colon \mathcal{C} \to \mathbb{R}$ satisfying

$$\Delta_{\bar{g}} v = 0 \quad \text{on } \mathcal{C}, \qquad v|_{M\times\{0\}} = u \quad \text{on } \partial\mathcal{C}.$$

If $u \equiv 1$, then $v \equiv 1$ is a solution, so $u \mapsto v$ *does not* map into $H^1(\mathcal{C})$ as constants have infinite $L^2(\mathcal{C})$ norm. Therefore, we need to work in the bigger space $X(\mathcal{C})$, defined in (4.6) as

$$X(\mathcal{C}) := \overline{H^1(\mathcal{C})}^{\|\cdot\|_{X(\mathcal{C})}} \text{ where } \|v\|_{X(\mathcal{C})}^2 := \|\nabla_{\bar{g}} v\|_{L^2(\mathcal{C})}^2 + \|\mathcal{T}v\|_{L^2(M)}^2 \quad \text{for } v \in H^1(\mathcal{C}).$$

Note that $H^1(\mathcal{C}) \overset{d}{\hookrightarrow} X(\mathcal{C})$.

Remark 4.2.2. The constant functions belong to $X(\mathcal{C})$ but not $H^1(\mathcal{C})$. To see this, take $u_n \in H^1(\mathcal{C})$ with

$$u_n(x,y) = \begin{cases} c & : y \in (0,n] \\ \frac{c}{n}(2n-y) & : y \in (n,2n] \\ 0 & : y \in (2n,\infty) \end{cases}$$

which satisfies $\nabla_M u_n = 0$ and $\partial_y u_n = -c/n\chi_{(n,2n)}$. Note that

$$\begin{aligned} \|u_n - u_m\|_{\tilde{X}(\mathcal{C})}^2 &= \|\partial_y(u_n - u_m)\|_{L^2(\mathcal{C})}^2 \\ &\le 2\left(\|\partial_y u_n\|_{L^2(\mathcal{C})}^2 + \|\partial_y u_m\|_{L^2(\mathcal{C})}^2\right) \\ &= 2\left(\int_n^{2n}\int_M \frac{c^2}{n^2} + \int_m^{2m}\int_M \frac{c^2}{m^2}\right) \\ &= 2|M|\left(\frac{c^2}{n} + \frac{c^2}{m}\right) \to 0 \end{aligned}$$

as n, $m \to \infty$. So $u := (u_n)$ is a Cauchy sequence in the $\tilde{X}(\mathcal{C})$ norm, and it follows that

$$\|u\|^2_{X(\mathcal{C})} = \lim_{n\to\infty} \int_n^{2n} \int_M \frac{c^2}{n^2} + \int_M c^2 = \lim_{n\to\infty} |M| \left(c^2 + \frac{c^2}{n}\right) = |M|c^2$$

using $\|u\|^2_{X(\mathcal{C})} := \lim_{n\to\infty} \|u_n\|^2_{\tilde{X}(\mathcal{C})}$. Then the constant c can be identified with u.

Lemma 4.2.3 (Extension of the gradient to $X(\mathcal{C})$)**.** The gradient $\nabla_{\bar{g}} \colon H^1(\mathcal{C}) \to L^2(\mathcal{C})$ extends to a bounded linear map $\overline{\nabla_{\bar{g}}} \colon X(\mathcal{C}) \to L^2(\mathcal{C})$ such that $\overline{\nabla_{\bar{g}}}|_{H^1(\mathcal{C})} = \nabla_{\bar{g}}$ and

$$\overline{\nabla_{\bar{g}}} v = \lim_{n\to\infty} \nabla_{\bar{g}} v_n$$

for $v_n \in H^1(\mathcal{C})$ such that $v_n \to v$ in $X(C)$.

Proof. We have that $\nabla_{\bar{g}} \colon H^1(\mathcal{C}) \to L^2(\mathcal{C})$ satisfies

$$\|\nabla_{\bar{g}} v\|^2_{L^2(\mathcal{C})} \leq \|\nabla_{\bar{g}} v\|^2_{L^2(\mathcal{C})} + \|\mathcal{T} v\|^2_{L^2(M)} = \|v\|^2_{X(\mathcal{C})}$$

for all $v \in H^1(\mathcal{C})$. Since $H^1(\mathcal{C})$ is dense in $X(\mathcal{C})$, Lemma 4.A.1 provides the result. □

Theorem 4.2.4. For every $u \in H(M)$, there exists a unique weak solution $\overline{\mathcal{E}}u \in X(\mathcal{C})$ to the harmonic extension problem (4.5) satisfying $(\overline{\mathcal{E}}u)(0) = u$ in $L^2(M)$ and

$$\int_{\mathcal{C}} \overline{\nabla_{\bar{g}}}(\overline{\mathcal{E}}u) \nabla_{\bar{g}} \eta = 0 \qquad \text{for all } \eta \in H^1(\mathcal{C}) \text{ with } \mathcal{T}\eta = 0.$$

When $\bar{u} = 0$, we write the solution as $\mathcal{E}u$ which satisfies $\frac{1}{|M|} \int_M (\mathcal{E}u)(y) = 0$ for all $y \in [0, \infty)$. The map $\overline{\mathcal{E}} \colon H(M) \to X(\mathcal{C})$ satisfies $\overline{\mathcal{E}}u = \mathcal{E}(u - \bar{u}) + \bar{u}$ and $\overline{\nabla_{\bar{g}}}(\overline{\mathcal{E}}u) = \nabla_{\bar{g}} \mathcal{E}(u - \bar{u})$. Furthermore, (if $\bar{u} = 0$)

$$\|\mathcal{E}u\|^2_{L^2(\mathcal{C})} \leq \frac{\|u\|^2_{L^2(M)}}{2\lambda_1^{1/2}} \tag{4.11}$$

$$\|\nabla_{\bar{g}} \mathcal{E}u\|^2_{L^2(\mathcal{C})} = \left\|(-\Delta_M)^{\frac{1}{4}} u\right\|^2_{L^2(M)} = |u|^2_{H(M)}. \tag{4.12}$$

Proof. The proof of the well-posedness is essentially the same as that of Theorem 2.1 in [117]. Suppose for now that $\bar{u} = 0$. Set

$$(\mathcal{E}u)(x, y) := v(x, y) := \sum_{k=1}^{\infty} e^{-y\lambda_k^{1/2}} (u, \varphi_k)_{L^2(M)} \varphi_k(x)$$

which is a sum that converges in $L^2(M)$ for each fixed $y \in [0, \infty)$, and we claim that this is a solution. Note that

$$\int_M |v(y)|^2 = \sum_{k\geq 1} e^{-2y\lambda_k^{1/2}} |(u, \varphi_k)_{L^2(M)}|^2$$

and also, using (4.10),

$$\int_M |v_y(y)|^2 = \int_M |\sum_{k\geq 1} \lambda_k^{1/2} e^{-y\lambda_k^{1/2}} (u, \varphi_k)_{L^2(M)} \varphi_k|^2 = \sum_{k\geq 1} \lambda_k e^{-2y\lambda_k^{1/2}} |(u, \varphi_k)_{L^2(M)}|^2,$$
$$\int_M |\nabla_M y|^2 = \sum_{k\geq 1} \lambda_k e^{-2y\lambda^{1/2}} |(u, \varphi_k)_{L^2(M)}|^2.$$

These expressions can be integrated over y since the sums converge uniformly, see Lemma 4.2.5. This implies that

$$\|v\|^2_{L^2(\mathcal{C})} = \sum_{k\geq 1} |(u, \varphi_k)_{L^2(M)}|^2 \int_0^\infty e^{-2y\lambda_k^{1/2}} = \sum_{k\geq 1} \frac{|(u, \varphi_k)_{L^2(M)}|^2}{2\lambda_k^{1/2}} \leq \frac{\|u\|^2_{L^2(M)}}{2\lambda_1^{1/2}},$$

and

$$\|\nabla_{\bar{g}} v\|^2_{L^2(\mathcal{C})} = 2 \sum_{k\geq 1} \lambda_k |(u, \varphi_k)_{L^2(M)}|^2 \int_0^\infty e^{-2y\lambda^{1/2}} = |u|^2_{H(M)}.$$

This proves properties (4.11) and (4.12).

Since the partial sums $v_N(y) = \sum_{k=1}^N e^{-y\lambda_k^{1/2}} (u, \varphi_k)_{L^2(M)} \varphi_k$ converge in $L^2(M)$ to $v(y)$, we have that

$$0 = (v_N(y), 1)_{L^2(M)} \to (v(y), 1)_{L^2(M)}$$

giving that $v(y)$ has mean value zero. Let $\eta \in H^1(\mathcal{C})$ with $\mathcal{T}\eta = 0 = \eta(x, 0)$. For almost all y, we have $\eta(x, y) = \sum_{k=0}^\infty (\eta(\cdot, y), \varphi_k)_{L^2(M)} \varphi_k(x)$; let us write the coefficients as $\eta_k(y)$. Now, we have

$$\int_M v_{yy}(y)\eta(y) = \sum_{k=1}^\infty \lambda_k e^{-y\lambda_k^{1/2}} (u, \varphi_k)_{L^2(M)} \eta_k(y)$$

and

$$\int_M \nabla_M v(y) \nabla_M \eta(y) = \sum_{k=1}^\infty \lambda_k e^{-y\lambda_k^{1/2}} (u, \varphi_k)_{L^2(M)} \eta_k(y) = \int_M v_{yy}(y)\eta(y) \tag{4.13}$$

using (4.10). Using integration by parts we get

$$\begin{aligned}\int_{\mathcal{C}} \nabla_M v \nabla_M \eta + v_y \eta_y &= \int_{\mathcal{C}} \nabla_M v \nabla_M \eta - v_{yy}\eta + \int_{\partial\mathcal{C}} v_y \eta \\ &= \int_M v_y(x,0)\eta(x,0) \qquad\qquad \text{(using (4.13))} \\ &= 0\end{aligned}$$

as η has trace zero. This proves that v is a weak solution. Uniqueness follows easily by assuming there are two solutions, taking the difference of the weak forms and then testing with the difference (which has trace zero).

Therefore, the map $\mathcal{E}\colon \{u \in H(M) \mid \bar{u} = 0\} \to H^1(\mathcal{C})$ is well-defined. Now suppose $\bar{u} \neq 0$. Define

$$\overline{\mathcal{E}}(u) := \mathcal{E}(u - \bar{u}) + \bar{u}.$$

Note that $\overline{\nabla_{\bar{g}}}(\overline{\mathcal{E}}u) = \overline{\nabla_{\bar{g}}}(\mathcal{E}(u-\bar{u}) + \bar{u}) = \nabla_{\bar{g}}\mathcal{E}(u-\bar{u}) + \overline{\nabla_{\bar{g}}}\bar{u}$ with the last equality by linearity and the fact that $\mathcal{E}(u-\bar{u}) \in H^1(\mathcal{C})$. We have $\overline{\nabla_{\bar{g}}}\bar{u} = \lim_{n\to\infty} \nabla_{\bar{g}} u_n$ where $u_n \in H^1(\mathcal{C})$ converges to $\bar{u}$ in $X(\mathcal{C})$, which is a constant. Let us choose u_n as in Remark 4.2.2 (with c in the remark replaced with $\bar{u}$), which tells us that $\lim_{n\to\infty} \nabla_{\bar{g}} u_n = \lim_{n\to\infty} -\frac{\bar{u}}{n}\chi_{(n,2n)} = 0$ since $\int_n^{2n}\int_M \bar{u}^2/n^2 = |M|\bar{u}^2/n \to 0$, i.e., $\nabla_{\bar{g}} u_n \to 0$ in $L^2(\mathcal{C})$. This proves that $\overline{\nabla_{\bar{g}}}(\overline{\mathcal{E}}u) = \nabla_{\bar{g}}\mathcal{E}(u-\bar{u})$. $\square$

Equality (4.12) shows that $\overline{\mathcal{E}}\colon H(M) \to X(\mathcal{C})$ is an isometry:

$$\left\|\overline{\mathcal{E}}u\right\|_{X(\mathcal{C})} = \|u\|_{H(M)}.$$

Lemma 4.2.5. The solution satisfies $\mathcal{E}u \in C^0([0,\infty); L^2(M)) \cap C^\infty((0,\infty); H^1(M))$ and the infinite sums that define $\|(\mathcal{E}u)(y)\|_{L^2(M)}$ and $\|\nabla_{\bar{g}}(\mathcal{E}u)(y)\|_{L^2(M)}$ are uniformly convergent on $[0,\infty)$ and $[\epsilon,\infty)$ respectively for any $\epsilon > 0$.

Proof. (1) That $\mathcal{E}u \in C^0([0,\infty); L^2(M))$ is easy to see since

$$\|v(y)\|^2_{L^2(M)} = \sum_{k=1}^{\infty} e^{-2y\sqrt{\lambda_k}} |(u,\varphi_k)|^2$$

and this is continuous as a function of y, because $e^{-2y\sqrt{\lambda_k}}|(u,\varphi_k)|^2 \leq |(u,\varphi_k)|^2$ gives uniform convergence of the sum by the Weierstrass M-test.

(2) Let us see now why $\mathcal{E}u \in C^0([\epsilon,\infty); H^1(M))$ for any $\epsilon > 0$. Note first of all that $v(y) \in H^1(M)$ if $y > 0$: the condition to check is (4.2.2), which translates

to requiring

$$\sum_{k\geq 1}(1+\lambda_k)e^{-2y\sqrt{\lambda_k}}|(u,\varphi_k)|^2 < \infty.$$

We have

$$\lambda_k e^{-2y\sqrt{\lambda_k}} = \frac{\sqrt{\lambda_k}}{y} y\sqrt{\lambda_k} e^{-2y\sqrt{\lambda_k}} \leq \frac{\sqrt{\lambda_k}}{y} e^{-y\sqrt{\lambda_k}} \leq \frac{\sqrt{\lambda_k}}{\epsilon} e^{-\epsilon\sqrt{\lambda_1}}$$

using $xe^{-2x} \leq e^{-x}$. This calculation implies that

$$\lambda_k e^{-2y\sqrt{\lambda_k}}|(u,\varphi_k)|^2 \leq \frac{e^{-\epsilon\sqrt{\lambda_1}}}{\epsilon}\sqrt{\lambda_k}|(u,\varphi_k)|^2 =: M_k$$

and $\sum_{k=1}^{\infty} M_k$ is finite and therefore by the Weierstrass M-test, the convergence of the sum

$$\sum_{k=1}^{\infty} \lambda_k e^{-2y\sqrt{\lambda_k}}|(u,\varphi_k)|^2 = \|\nabla_M v(y)\|^2_{L^2(M)}$$

is uniform, with the equality by using (4.10) (which is valid since we showed above that $v(y) \in H^1(M)$). Therefore $y \mapsto \|v(y)\|_{H^1(M)}$ is continuous on $[\epsilon,\infty)$. This shows that $v \in C^0((0,\infty); H^1(M))$.

(3) A similar argument enables us to prove that $\mathcal{E}u \in C^\infty((0,\infty); H^1(M))$ with

$$\left\|v^{(m)}(y)\right\|^2_{H^1(M)} = \sum_{k=1}^{\infty}(1+\lambda_k)\lambda_k^m e^{-2y\sqrt{\lambda_k}}|(u,\varphi_k)|^2$$

by making use of $xe^{-2x} \leq e^{-x}$ iteratively. In fact, we obtain for any $m \in \mathbb{N}$ and $l \in \mathbb{N}$ with $l < m$:

$$\lambda_k^m e^{-2y\sqrt{\lambda_k}} \leq \frac{1}{y^{2l+1}} \underbrace{2^{2l}\cdot 2^{2l-1}\cdot \ldots . \cdot 2}_{=:C(l)} \lambda_k^{m-l-\frac{1}{2}} e^{-2^{-2l}y\lambda_k^{1/2}}.$$

So in particular, with $l = m-1$ and if $y \geq \epsilon > 0$:

$$\lambda_k^m e^{-2y\sqrt{\lambda_k}} \leq \frac{e^{-2^{-2(m-1)}\epsilon\lambda_1^{1/2}}C(m-1)}{\epsilon^{2m-1}}\lambda_k^{1/2}.$$

Now we can use the Weierstrass M-test again to obtain the uniform convergence of the partial sums

$$\sum_{k=1}^{N}(1+\lambda_k)\lambda_k^m e^{-2y\sqrt{\lambda_k}}|(u,\varphi_k)|^2.$$

$\square$

Regarding the next lemma, see also [28].

Lemma 4.2.6. The fractional Laplacian of $u \in H(M)$ is recovered through the Dirichlet-to-Neumann map:

$$(-\Delta_M)^{1/2}u = -\lim_{y\to 0^+} \frac{\partial \overline{\mathcal{E}}u}{\partial y} \qquad \text{in } H(M)^*.$$

Proof. If $\bar{u} = 0$ and $\eta \in H(M)$,

$$-\langle v_y(y), \eta\rangle_{H(M)^*,H(M)} = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e^{-y\sqrt{\lambda_k}} (u, \varphi_k)_{L^2(M)} (\eta, \varphi_k)_{L^2(M)}$$

and the sum uniformly converges by Abel's test, because $e^{-y\sqrt{\lambda_k}}$ is decreasing and bounded, and

$$\sum_{k=1}^{\infty} \sqrt{\lambda_k} (u, \varphi_k)_{L^2(M)} (\eta, \varphi_k)_{L^2(M)} = \langle (-\Delta_M)^{1/2} u, \eta\rangle_{H(M)^*,H(M)} < \infty.$$

Therefore, we can take the limit:

$$\begin{aligned}\lim_{y\to 0^+} -\langle v_y(y), \eta\rangle_{H(M)^*,H(M)} &= \sum_{k=1}^{\infty} \sqrt{\lambda_k} (u, \varphi_k)_{L^2(M)} (\eta, \varphi_k)_{L^2(M)} \\ &= \langle (-\Delta_M)^{1/2} u, \eta\rangle_{H(M)^*,H(M)}.\end{aligned}$$

If $\bar{u} \neq 0$,

$$(-\Delta_M)^{1/2}u = (-\Delta_M)^{1/2}(u - \bar{u}) = -\lim_{y\to 0^+} \partial_y(\mathcal{E}(u - \bar{u})) = -\lim_{y\to 0^+} \partial_y \overline{\mathcal{E}}u.$$

□

The following cut-off function will be of use here and in §4.6.

Definition 4.2.7 (Cut-off function)**.** For any $\rho > 0$, there exists a smooth cut-off function ψ_ρ such that

$$\psi_\rho(y) = \begin{cases} 1 & : y \in [0, \rho] \\ 0 & : y \in [2\rho, \infty) \end{cases}$$

and $-\frac{1}{\rho} C\sqrt{\psi(1 - \frac{y}{\rho})} \leq \psi_\rho'(y) \leq 0$ on $[\rho, 2\rho]$, with C not depending on ρ. It follows that $\psi_\rho(y) \to 1$ pointwise, $|\psi_\rho(y)| \leq 1$, $\psi_\rho'(y) \to 0$ pointwise and $|\psi_\rho'(y)| \leq \frac{1}{\rho} C\sqrt{\psi(1 - \frac{y}{\rho})} \leq C$ (for $\rho \geq 1$) on $[\rho, 2\rho]$.

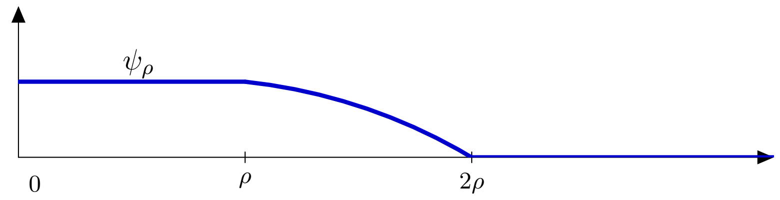

Figure 4.2.1: A sketch of the cut-off function ψ_ρ

Define a map $\mathcal{N}\colon H^{1/2}(M) \to H^{-1/2}(M)$ by

$$\langle \mathcal{N}u, h\rangle_{H^{-1/2}(M),H^{1/2}(M)} := \int_{\mathcal{C}} \overline{\nabla_{\bar g}}(\overline{\mathcal{E}}u)\nabla_{\bar g}\tilde h$$

where $\tilde h \in H^1(\mathcal{C})$ is any extension of h. This map is well-defined since if we had two arbitrary extensions $\tilde h_1$ and $\tilde h_2$, then

$$\int_{\mathcal{C}} \overline{\nabla_{\bar g}}(\overline{\mathcal{E}}u)\nabla_{\bar g}\tilde h_1 - \int_{\mathcal{C}} \overline{\nabla_{\bar g}}(\overline{\mathcal{E}}u)\nabla_{\bar g}\tilde h_2 = \int_{\mathcal{C}} \overline{\nabla_{\bar g}}(\overline{\mathcal{E}}u)\nabla_{\bar g}(\tilde h_1 - \tilde h_2) = 0$$

by definition of the weak formulation that $\overline{\mathcal{E}}u$ satisfies, since $\mathcal{T}(\tilde h_1 - \tilde h_2) = 0$. The fact that the extension can be arbitrary will be extremely useful later on. Furthermore, by taking the extension of h to be $\mathcal{E}(h-\bar h) + \psi_\rho \bar h \in H^1(\mathcal{C})$, one can see that $\mathcal{N}u$ is linear, and by the calculation

$$\begin{aligned}
|\langle \mathcal{N}u, h\rangle_{H^{-1/2}(M),H^{1/2}(M)}| &\le \int_{\mathcal{C}} |\overline{\nabla_{\bar g}\mathcal{E}}u||\nabla_{\bar g}(\mathcal{E}(h-\bar h) + \psi_\rho \bar h)| \\
&\le |u-\bar u|_{H(M)} \left(\left|h-\bar h\right|_{H(M)} + |\bar h| \left\|\psi_\rho'\right\|_{L^2(\mathcal{C})}\right) \\
&\le |u|_{H(M)} \left(|h|_{H(M)} + C_1 \|h\|_{L^2(M)} \left\|\psi_\rho'\right\|_{L^2(\mathcal{C})}\right) \\
&\le C_2 |u|_{H(M)} \|h\|_{H(M)} \\
&\le C_3 |u|_{H(M)} \|h\|_{H^{1/2}(M)},
\end{aligned}$$

$\mathcal{N}u$ is indeed in the dual space of $H^{1/2}(M)$. We can write $\mathcal{N}u = \frac{\partial \overline{\mathcal{E}}u}{\partial \nu}\big|_{y=0}$, i.e., $\mathcal{N}$ is the Dirichlet-to-Neumann map; this notation is justified since, if for example $\Delta_{\bar g}\mathcal{E}(u) \in L^2(\mathcal{C})$, the standard Green's formula implies $\int_{\partial\mathcal{C}} \frac{\partial \mathcal{E}u}{\partial \nu} w = \int_{\mathcal{C}} w\Delta_{\bar g}\mathcal{E}u + \int_{\mathcal{C}} \nabla_{\bar g}\mathcal{E}u\nabla w = \int_{\mathcal{C}} \nabla_{\bar g}\mathcal{E}u\nabla w$.

Lemma 4.2.8. Let $u \in H(M)$ with $\bar u = 0$. The harmonic extension $\mathcal{E}u$ is the unique minimiser of the energy

$$J(v) = \frac{1}{2}\int_{\mathcal{C}} |\nabla_{\bar g} v|^2$$

over the set $H^1_u(\mathcal{C}) := \{v \in H^1(\mathcal{C}) \mid \mathcal{T}v = u\}$.

Proof. With $w \in H^1_u(\mathcal{C})$, test the weak form $\mathcal{E}u = v$ satisfies with $\eta = v - w$:

$$\int_{\mathcal{C}} |\nabla_{\bar{g}} v|^2 - \nabla_{\bar{g}} v \nabla_{\bar{g}} w = 0$$

which gives

$$J(v) = \frac{1}{2}\int_{\mathcal{C}} |\nabla_{\bar{g}} v|^2 \leq \frac{1}{4}\|\nabla_{\bar{g}} v\|^2_{L^2(\mathcal{C})} + \frac{1}{4}\|\nabla_{\bar{g}} w\|^2_{L^2(\mathcal{C})},$$

and rearranging shows $J(v) \leq J(w)$. Uniqueness follows since J is strictly convex. $\square$

Lemma 4.2.9. The space $H(M) = H^{1/2}(M)$. This means that $H(M) \subset H^{1/2}(M)$ and $H^{1/2}(M) \subset H(M)$ with an equivalence of norms (the constants in the equivalence of norms will depend on M).

Proof. Given $u \in H(M)$ with $\bar{u} = 0$, define $v = \mathcal{E}u$, which we know belongs to $H^1(\mathcal{C})$ from Theorem 4.2.4 and so $\mathcal{T}v = u \in H^{1/2}(M)$ since $\mathcal{T}$ has range in $H^{1/2}(M)$. For the case $\bar{u} \neq 0$, we have that $u - \bar{u} \in H^{1/2}(M)$, which implies that $u = u - \bar{u} + \bar{u} \in H^{1/2}(M)$, since the constants are elements of $H^1(M) \subset H^{1/2}(M)$.

Now we prove the reverse inclusion. Recall that a function $u \in L^2(M) + H^1(M)$ belongs to the interpolation space $H^{1/2}(M)$ as defined by the K-method if, given the K-functional (for $t > 0$)

$$K(t,u) = \inf_{\substack{u = u_0 + u_1 \\ u_0 \in L^2(M) \\ u_1 \in H^1(M)}} \left(\|u_0\|^2_{L^2(M)} + t^2\|u_1\|^2_{H^1(M)}\right)^{1/2},$$

the following norm is finite:

$$\|u\|_{H^{1/2}(M)} = \left(\int_0^\infty (t^{-1/2}K(t,u))^2 \frac{\mathrm{d}t}{t}\right)^{1/2}.$$

See [86, Chapter 1, §15], [25, Appendix B], [89, Appendix B] for more information. We follow the ideas of the proof of Theorem B.2 in [25] now. Let $u \in H^{1/2}(M)$, and write $u = \sum_{k=0}^{\infty} u_k \varphi_k$ and take $v = \sum_{k=0}^{\infty} v_k \varphi_k \in H^1(M)$. Then

$$\begin{aligned} K^2(t,u) &= \inf_{v \in H^1(M)} \left(\|u - v\|_{L^2(M)}^2 + t^2 \|v\|_{H^1(M)}^2 \right) \\ &= \inf_{v \in H^1(M)} \left(\sum_{k=0}^{\infty} |u_k - v_k|^2 + t^2 \sum_{k=0}^{\infty} (1 + \lambda_k) |v_k|^2 \right), \end{aligned}$$

and the expression in the infimum is minimised when $v_k = x$ satisfies

$$\frac{d}{dx} |u_k - x|^2 + t^2 (1 + \lambda_k) |x|^2 = 2t^2 (1 + \lambda_k) x + 2(x - u_k) = 0$$

i.e., when $v_k = u_k / (1 + t^2(1 + \lambda_k))$, so that

$$K^2(t,u) = \sum_{k=0}^{\infty} \frac{t^2(1+\lambda_k)}{1 + t^2(1+\lambda_k)} |u_k|^2.$$

Therefore,

$$\begin{aligned} \|u\|_{H^{1/2}(M)}^2 &= \int_0^{\infty} \sum_{k=0}^{\infty} \frac{(1+\lambda_k)}{1 + t^2(1+\lambda_k)} |u_k|^2 \mathrm{d}t && (4.14) \\ &= \sum_{k=0}^{\infty} (1+\lambda_k) |u_k|^2 \int_0^{\infty} \frac{1}{1 + t^2(1+\lambda_k)} \mathrm{d}t \\ &= \frac{\pi}{2} \sum_{k=0}^{\infty} \sqrt{1+\lambda_k} |u_k|^2 && (4.15) \\ &\geq \frac{\pi}{2} \sum_{k=0}^{\infty} \sqrt{\lambda_k} |u_k|^2 \\ &= \frac{\pi}{2} |u|_{H(M)}^2 . \end{aligned}$$

The integration term by term is justified for the following reason. On $[\epsilon, \infty)$, we have

$$\frac{(1+\lambda_k)}{1 + t^2(1+\lambda_k)} |u_k|^2 \leq \frac{(1+\lambda_k)}{t^2(1+\lambda_k)} |u_k|^2 \leq \frac{1}{\epsilon^2} |u_k|^2$$

and the right hand side is convergent. Therefore the sum (4.14) is uniformly convergent on $[\epsilon, \infty)$. Then one can integrate over this interval and send $\epsilon \to 0$, using continuity and the monotone convergence theorem.

The above calculation implies that $\|u\|_{H(M)}^2 \leq \pi^{-1}(2+\pi) \|u\|_{H^{1/2}(M)}^2$. From (4.15), using $\sqrt{1+\lambda_k} \leq 1 + \sqrt{\lambda_k}$, we have that $\|u\|_{H^{1/2}(M)}^2 \leq 2^{-1}\pi \|u\|_{H(M)}^2$. □

We could also have proved this lemma via the J-method of interpolation and Weyl's law [76, Chapter 3, equation (3.2.24)], as is done in [19, §3.1.3] on a bounded domain. Another approach, relying explicitly on the Gagliardo norm on $H^{1/2}(M)$ when M is a hypersurface, can also work with the use of two-sided Gaussian estimates on the heat kernel, similar to [117, §2.2] for the case of the Neumann Laplacian in a bounded domain.

The trace map can be extended to the space $X(\mathcal{C})$ (cf. [117, Lemma 2.4]).

Lemma 4.2.10. There exists a bounded linear trace map $\overline{\mathcal{T}}\colon X(\mathcal{C}) \to H(M)$ such that

$$\left\|\overline{\mathcal{T}}v\right\|_{H(M)} \leq \|v\|_{X(\mathcal{C})} \quad \text{for } v \in X(\mathcal{C}),$$

$\overline{\mathcal{T}}w = \mathcal{T}w$ for $w \in H^1(\mathcal{C}) \subset X(\mathcal{C})$ (i.e., $\overline{\mathcal{T}}$ extends $\mathcal{T}$), and $\overline{\mathcal{T}}w := \lim_{n\to\infty} \mathcal{T}w_n$ for $w_n \in H^1(\mathcal{C})$ converging to w in $X(\mathcal{C})$.

Proof. Let $w \in H^1(\mathcal{C})$ be arbitrary with $\mathcal{T}w =: w_0$. It follows that if $\overline{w_0} = 0$, using (4.12),

$$\left\|(-\Delta_M)^{\frac{1}{4}} w_0\right\|^2_{L^2(M)} = \|\nabla_{\bar{g}}\mathcal{E}w_0\|^2_{L^2(\mathcal{C})} = 2J(\mathcal{E}w_0) \leq 2J(w) = \|\nabla_{\bar{g}}w\|^2_{L^2(\mathcal{C})},$$

since the harmonic extension minimises J. Since this inequality does not see constants, we can drop the assumption $\overline{w_0} = 0$. Then

$$\|w_0\|^2_{L^2(M)} + \left\|(-\Delta_M)^{\frac{1}{4}} w_0\right\|^2_{L^2(M)} \leq \|\nabla_{\bar{g}}w\|^2_{L^2(\mathcal{C})} + \|w_0\|^2_{L^2(M)} = \|w\|^2_{X(\mathcal{C})}.$$

That is, $\mathcal{T}\colon H^1(\mathcal{C}) \to H(M)$ satisfies $\|\mathcal{T}w\|_{H(M)} \leq \|w\|_{X(\mathcal{C})}$. Then Lemma 4.A.1 gives the result. $\square$

Remark 4.2.11. The trace map $\overline{\mathcal{T}}$ between $\{v \in X(\mathcal{C}) \mid \Delta_{\bar{g}}v = 0\}$ and $H^{1/2}(M)$ is invertible with inverse $\overline{\mathcal{E}}\colon H^{1/2}(M) \to X(\mathcal{C})$.

The following lemma is a seminorm boundedness property of the trace map; note the Gagliardo seminorm on the left hand side (the proof of Lemma 4.2.10 had the $H(M)$ seminorm on the left hand side instead).

Lemma 4.2.12. Let $M = \Gamma$ be a hypersurface of class C^1. For every $v \in H^1(\mathcal{C})$,

$$|\mathcal{T}v|_{W^{1/2,2}(\Gamma)} \leq C\,\|\nabla_{\bar{g}}v\|_{L^2(\mathcal{C})}.$$

Proof. Let $v \in H^1(\mathcal{C})$ satisfy $\frac{1}{|\Gamma|}\int_\Gamma v(y) = 0$ for all y. Using the trace theorem, we calculate

$$
\begin{aligned}
\|\mathcal{T}v\|_{W^{1/2,2}(\Gamma)} &\leq C_1(\|v\|_{L^2(\mathcal{C})} + \|\nabla_{\bar{g}} v\|_{L^2(\mathcal{C})}) \\
&= C_1 \left(\left(\int_0^\infty \int_\Gamma |v(y)|^2 \right)^{1/2} + \|\nabla_{\bar{g}} v\|_{L^2(\mathcal{C})} \right) \\
&\leq C_2 \left(\left(\int_0^\infty \int_\Gamma |\nabla_\Gamma v(y)|^2 \right)^{1/2} + \|\nabla_{\bar{g}} v\|_{L^2(\mathcal{C})} \right) \\
&\leq C_3 \|\nabla_{\bar{g}} v\|_{L^2(\mathcal{C})}
\end{aligned}
$$

where we used Poincaré's inequality, since $v(y)$ has mean value zero. Now suppose that $v \in H^1(\mathcal{C})$ does have not have spatial mean value zero for a.a. y. Then define

$$
\hat{v}(x,y) = v(x,y) - \frac{1}{|\Gamma|} \int_\Gamma v(y)
$$

which satisfies $\frac{1}{|\Gamma|} \int_\Gamma \hat{v}(y) = 0$ and $\hat{v} \in H^1(\mathcal{C})$ by Lemma 4.2.1. Then

$$
\begin{aligned}
\|\mathcal{T}\hat{v}\|_{W^{1/2,2}(\Gamma)} &\leq C_3 \left\| \nabla_{\bar{g}} \left(v - \frac{1}{|\Gamma|} \int_\Gamma v(y) \right) \right\|_{L^2(\mathcal{C})} \\
&\leq C_3 \left(\|\nabla_{\bar{g}} v\|_{L^2(\mathcal{C})} + \left\| \frac{1}{|\Gamma|} \partial_y \int_\Gamma v(y) \right\|_{L^2(\mathcal{C})} \right) \\
&= C_3 \left(\|\nabla_{\bar{g}} v\|_{L^2(\mathcal{C})} + \left\| \frac{1}{|\Gamma|} \int_\Gamma \partial_y v(y) \right\|_{L^2(\mathcal{C})} \right) \\
&\leq C_4 \|\nabla_{\bar{g}} v\|_{L^2(\mathcal{C})}
\end{aligned}
$$

but the left hand side is

$$
\begin{aligned}
\|\mathcal{T}\hat{v}\|_{W^{1/2,2}(\Gamma)} &= \left\| \mathcal{T}v - \frac{1}{|\Gamma|} \int_\Gamma v(0) \right\|_{W^{1/2,2}(\Gamma)} && \text{(using Lemma 4.2.1)} \\
&= \left(\left\| \mathcal{T}v - \frac{1}{|\Gamma|} \int_\Gamma v(0) \right\|_{L^2(\Gamma)}^2 + |\mathcal{T}v|_{W^{1/2,2}(\Gamma)}^2 \right)^{1/2} \\
& && \text{(because the seminorm does not see constants)} \\
&\geq |\mathcal{T}v|_{W^{1/2,2}(\Gamma)} .
\end{aligned}
$$

□

4.2.4 The truncated harmonic extension problem

Define $H_0^1(\mathcal{C}_R) := \{\eta \in H^1(\mathcal{C}_R) \mid \mathcal{T}_{R,y=0}\eta = \mathcal{T}_{R,y=R}\eta = 0\}$; this is a Hilbert space because it is a closed linear subspace of $H^1(\mathcal{C}_R)$.

Theorem 4.2.13. For every $u \in H(M)$, there exists a unique weak solution $v = \overline{\mathcal{E}}_R u \in H^1(\mathcal{C}_R)$ to the truncated harmonic extension problem (4.8)

$$\Delta_{\bar{g}} v_R = 0 \quad \text{on } \mathcal{C}_R := M \times [0,R], \qquad v_R|_{M\times\{0\}} = u, \qquad v_R|_{M\times\{R\}} = 0,$$

satisfying $(\overline{\mathcal{E}}_R u)(0) = u$ and $(\overline{\mathcal{E}}_R u)(R) = 0$ in $L^2(M)$ and

$$\int_{\mathcal{C}_R} \nabla_{\bar{g}} \overline{\mathcal{E}}_R u \nabla_{\bar{g}} \eta = 0 \quad \text{for all } \eta \in H_0^1(\mathcal{C}_R).$$

When $\bar{u} = 0$, we write the solution as $\mathcal{E}_R u$ which satisfies $\frac{1}{|M|}\int_M (\mathcal{E}_R u)(y) = 0$ for all $y \in [0,R]$. The map $\overline{\mathcal{E}}_R \colon H(M) \to H^1(\mathcal{C}_R)$ satisfies $\overline{\mathcal{E}}_R u = \mathcal{E}_R(u - \bar{u}) + \frac{R-y}{R}\bar{u}$.

Proof. Suppose that $\bar{u} = 0$ and define

$$(\mathcal{E}_R u)(x,y) := v(x,y) := \sum_{k=1}^{\infty} \left(\alpha_1(k,R) e^{\sqrt{\lambda_k} y} + \alpha_2(k,R) e^{-\sqrt{\lambda_k} y} \right) (u, \varphi_k)_{L^2(M)} \varphi_k(x)$$

where

$$\alpha_1(k,R) = -\frac{e^{-\sqrt{\lambda_k}R}}{e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R}} \quad \text{and} \quad \alpha_2(k,R) = \frac{e^{\sqrt{\lambda_k}R}}{e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R}}.$$

The formula for the solution comes from separation of variables and the infinite sum converges in $L^2(M)$ for all $y \in [0,R]$. To see that it is a weak solution, take a test function $\eta \in H_0^1(\mathcal{C}_R)$ with $\eta(x,y) = \sum_{k=0}^{\infty} (\eta(y), \varphi_k)\varphi_k(x)$ and calculate (using (4.10))

$$\begin{aligned}
&\int_M \nabla_\Gamma v(y) \nabla_\Gamma \eta(y) \\
&\quad = \sum_{k=1}^{\infty} \lambda_k \left(\alpha_1(k,R) e^{\sqrt{\lambda_k} y} + \alpha_2(k,R) e^{-\sqrt{\lambda_k} y} \right) (\eta(y), \varphi_k)_{L^2(M)} (u, \varphi_k)_{L^2(M)},
\end{aligned}$$

and since

$$v_{yy}(y) = \sum_{k=1}^{\infty} \lambda_k \left(\alpha_1(k,R) e^{\sqrt{\lambda_k} y} + \alpha_2(k,R) e^{-\sqrt{\lambda_k} y} \right) (u, \varphi_k)_{L^2(M)} \varphi_k,$$

we have

$$\int_M v_{yy}(y)\eta(y) = \int_M \nabla_\Gamma v \nabla_\Gamma \eta.$$

Then

$$\int_{\mathcal{C}_R} \nabla_\Gamma v \nabla_\Gamma \eta = \int_{\mathcal{C}_R} v_{yy}\eta = -\int_{C_R} v_y\eta_y + \int_{\partial\mathcal{C}_R} v_y\eta = -\int_{C_R} v_y\eta_y$$

with the last equality since η vanishes on the boundary; this implies the result. For u with mean value non-zero, we set $\overline{\mathcal{E}}_R u := \mathcal{E}_R(u-\bar{u}) + \frac{R-y}{R}\bar{u}$. This is a solution because

$$\int_{\mathcal{C}_R} \nabla_{\bar{g}}\left(\frac{R-y}{R}\bar{u}\right)\nabla_{\bar{g}}\eta = -\frac{\bar{u}}{R}\int_{\mathcal{C}_R}\partial_y\eta = -\frac{\bar{u}}{R}\int_0^R \frac{d}{dy}\int_M \eta = -\frac{\bar{u}}{R}\int_M (\eta(R)-\eta(0))$$

which equals zero. Lemmas 4.2.16 and 4.2.17 give that $\overline{\mathcal{E}}_R u$ is in $H^1(\mathcal{C}_R)$. □

Remark 4.2.14. Recall that

$$v(x,y) = \sum_{k=1}^\infty \left(\frac{e^{\sqrt{\lambda_k}(R-y)} - e^{-\sqrt{\lambda_k}(R-y)}}{e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R}}\right)(u,\varphi_k)_{L^2(M)}\varphi_k(x).$$

Set

$$f_{r,R}(n) = \frac{e^{nr}-e^{-nr}}{e^{nR}-e^{-nR}}$$

and observe that

$$\begin{aligned}
\frac{d}{dn}f_{r,R}(n) &= \frac{r(e^{nr}+e^{-nr})}{e^{nR}-e^{-nR}} - \frac{R(e^{nR}+e^{-nR})(e^{nr}-e^{-nr})}{(e^{nR}-e^{-nR})^2} \\
&= \frac{(e^{nr}-e^{-nr})}{(e^{nR}-e^{-nR})}\left(\frac{r(e^{nr}+e^{-nr})}{(e^{nr}-e^{-nr})} - \frac{R(e^{nR}+e^{-nR})}{(e^{nR}-e^{-nR})}\right) \\
&= \frac{(e^{nr}-e^{-nr})}{(e^{nR}-e^{-nR})}\left(r\coth(nr) - R\coth(nR)\right),
\end{aligned}$$

Define $g_n(s) = s\coth(ns)$ and note that $\frac{d}{ds}g_n(s) = \coth(ns) + ns(1-\coth^2(ns)) \geq 0$ whenever $ns \geq 0$ so that $g_n(r) - g_n(R) \leq 0$ and

$$\frac{d}{dn}f_{r,R}(n) \leq 0$$

since the factor in front of the g_n terms is positive. This implies that

$$k \mapsto \frac{e^{\sqrt{\lambda_k}(R-y)} - e^{-\sqrt{\lambda_k}(R-y)}}{e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R}} \quad \text{is monotone decreasing.}$$

Furthermore, as $r \mapsto \sinh(r)$ is increasing,

$$\frac{e^{\sqrt{\lambda_k}(R-y)} - e^{-\sqrt{\lambda_k}(R-y)}}{e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R}} = \frac{\sinh(\sqrt{\lambda_k}(R-y))}{\sinh(\sqrt{\lambda_k}R)} \leq 1.$$

Lemma 4.2.15. The solution satisfies $\mathcal{E}_R u \in C^0([0,R]; L^2(M)) \cap C^0((0,R]; H^1(M))$ and $\partial_y \mathcal{E}_R u \in C^0((0,R]; L^2(M))$. The infinite sums that define $\|(\mathcal{E}_R u)(y)\|_{L^2(M)}$ and $\|\nabla_{\bar{g}}(\mathcal{E}_R u)(y)\|_{L^2(M)}$ are uniformly convergent on $[0,R]$ and $[\epsilon, R]$ respectively, for any $\epsilon > 0$.

Proof. (1) We have

$$\|v(y)\|^2_{L^2(M)} = \sum_{k=1}^{\infty} \left(\frac{e^{\sqrt{\lambda_k}(R-y)} - e^{-\sqrt{\lambda_k}(R-y)}}{e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R}} \right)^2 |(u, \varphi_k)_{L^2(M)}|^2$$

and the sum is uniformly convergent by the Weierstrass M-test (the coefficient is bounded above by 1) so $\mathcal{E}_R u \in C^0([0,R]; L^2(M))$.

(2) We have

$$\int_M |\nabla_M v(y)|^2 = \sum_{k=1}^{\infty} \lambda_k \left(\alpha_1(k,R) e^{\sqrt{\lambda_k}y} + \alpha_2(k,R) e^{-\sqrt{\lambda_k}y} \right)^2 |(u, \varphi_k)_{L^2}|^2$$

and

$$\int_M |v_y(x,y)|^2 = \sum_{k=1}^{\infty} \lambda_k \left(\alpha_1(k,R) e^{\sqrt{\lambda_k}y} - \alpha_2(k,R) e^{-\sqrt{\lambda_k}y} \right)^2 |(u, \varphi_k)_{L^2}|^2,$$

which implies

$$\int_M |\nabla_{\bar{g}} v|^2 = 2 \sum_{k=1}^{\infty} \lambda_k \left(\alpha_1(k,R)^2 e^{2\sqrt{\lambda_k}y} + \alpha_2(k,R)^2 e^{-2\sqrt{\lambda_k}y} \right) |(u, \varphi_k)_{L^2}|^2.$$

Let us define

$$\begin{aligned} a_k(y) &:= \alpha_1(k,R)^2 e^{2\sqrt{\lambda_k}y} + \alpha_2(k,R)^2 e^{-2\sqrt{\lambda_k}y} \\ &= \frac{e^{2\sqrt{\lambda_k}(R-y)} + e^{2\sqrt{\lambda_k}(y-R)}}{(e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R})^2} \\ &= \frac{\cosh(2\sqrt{\lambda_k}(R-y))}{2\sinh^2(\sqrt{\lambda_k}R)}. \end{aligned}$$

Let $y \geq \epsilon > 0$. The function a_k is largest when $y = \epsilon$, so we have

$$
\begin{aligned}
\lambda_k a_k(y) &\leq \lambda_k \frac{e^{2\sqrt{\lambda_k}(R-\epsilon)} + e^{2\sqrt{\lambda_k}(\epsilon-R)}}{(e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R})^2} \\
&= \lambda_k e^{-2\sqrt{\lambda_k}\epsilon} \frac{e^{2\sqrt{\lambda_k}R}}{(e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R})^2} + \lambda_k \frac{e^{-2\sqrt{\lambda_k}(R-\epsilon)}}{(e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R})^2} \\
&\leq \frac{\sqrt{\lambda_k}}{\epsilon} \sqrt{\lambda_k}\epsilon e^{-2\sqrt{\lambda_k}\epsilon} C_1(\lambda_1, R) + \frac{\sqrt{\lambda_k}}{R-\epsilon} \frac{\sqrt{\lambda_k}(R-\epsilon) e^{-2\sqrt{\lambda_k}(R-\epsilon)}}{(e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R})^2} \\
&\qquad\qquad \left(\text{because } C_1(\lambda_k, R) := \tfrac{e^{2\sqrt{\lambda_k}R}}{(e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R})^2} \text{ is decreasing in } k\right) \\
&\leq \frac{\sqrt{\lambda_k}}{\epsilon} e^{-\sqrt{\lambda_k}\epsilon} C_1(\lambda_1, R) + \frac{\sqrt{\lambda_k}}{R-\epsilon} \frac{e^{-\sqrt{\lambda_k}(R-\epsilon)}}{(e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R})^2} \\
&\leq \frac{\sqrt{\lambda_k}}{\epsilon} e^{-\sqrt{\lambda_1}\epsilon} C_1(\lambda_1, R) + \frac{\sqrt{\lambda_k}}{R-\epsilon} e^{-\sqrt{\lambda_1}(R-\epsilon)} C_2(\lambda_1, R). \\
&\qquad\qquad\qquad\qquad \text{(again because of monotonicity)}
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\lambda_k a_k(y) |(u, \varphi_k)_{L^2(M)}|^2 \\
&\leq \left(\frac{1}{\epsilon} e^{-\sqrt{\lambda_k}\epsilon} C_1(\lambda_1, R) + \frac{1}{R-\epsilon} e^{-\sqrt{\lambda_1}(R-\epsilon)} C_2(\lambda_1, R) \right) \sqrt{\lambda_k} |(u, \varphi_k)_{L^2(M)}|^2
\end{aligned}
$$

and the right hand side is summable, so by the Weierstrass M-test, so the sum defining $\|\nabla_{\bar{g}} v(y)\|_{L^2(M)}$ is uniformly convergent over $[\epsilon, R]$. □

Lemma 4.2.16. For all $u \in H(M)$,

$$
\left\|\overline{\mathcal{E}}_R u\right\|^2_{L^2(\mathcal{C}_R)} \leq \frac{1}{2\sqrt{\lambda_1}} \|u - \bar{u}\|^2_{L^2(M)} + 4R|M||\bar{u}|^2.
$$

Proof. First let $\bar{u} = 0$ and set $v = \overline{\mathcal{E}}_R u$. Start with

$$
\int_M |v(y)|^2 = \sum_{k=1}^{\infty} \left(\alpha_1(k,R) e^{\sqrt{\lambda_k} y} + \alpha_2(k,R) e^{-\sqrt{\lambda_k} y} \right)^2 |(u, \varphi_k)|^2
$$

and observe that

$$
\begin{aligned}
\left(\alpha_1(k,R) e^{\sqrt{\lambda_k} y} + \alpha_2(k,R) e^{-\sqrt{\lambda_k} y} \right)^2 &= \frac{(e^{\sqrt{\lambda_k}(R-y)} - e^{\sqrt{\lambda_k}(y-R)})^2}{(e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R})^2} \\
&= \frac{e^{2\sqrt{\lambda_k}(R-y)} + e^{2\sqrt{\lambda_k}(y-R)} - 2}{(e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R})^2},
\end{aligned}
$$

and thus

$$
\begin{aligned}
\int_0^R \left(\alpha_1(k,R)e^{\sqrt{\lambda_k}y} + \alpha_2(k,R)e^{-\sqrt{\lambda_k}y}\right)^2 &= \frac{[\frac{e^{2\sqrt{\lambda_k}(R-y)}}{-2\sqrt{\lambda_k}}]_0^R + [\frac{e^{2\sqrt{\lambda_k}(y-R)}}{2\sqrt{\lambda_k}}]_0^R - 2R}{(e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R})^2} \\
&= \frac{e^{2\sqrt{\lambda_k}R} - 1 + 1 - e^{-2\sqrt{\lambda_k}R} - 4R\sqrt{\lambda_k}}{2\sqrt{\lambda_k}(e^{\sqrt{\lambda_k}R} - e^{-\sqrt{\lambda_k}R})^2} \\
&= \frac{1}{2\sqrt{\lambda_k}} \frac{e^{2\sqrt{\lambda_k}R} - e^{-2\sqrt{\lambda_k}R} - 4R\sqrt{\lambda_k}}{e^{2\sqrt{\lambda_k}R} + e^{-2\sqrt{\lambda_k}R} - 2} \\
&\le \frac{1}{2\sqrt{\lambda_k}}.
\end{aligned}
$$

This then implies (with the term by term integration possible by the uniform convergence given in Lemma 4.2.15)

$$
\int_0^R \int_M |v|^2 \le \sum_{k=1}^{\infty} \frac{1}{2\sqrt{\lambda_k}} |(u,\varphi_k)|^2 \le \frac{1}{2\sqrt{\lambda_1}} \|u\|^2_{L^2(M)}.
$$

If $\bar{u} \neq 0$, noting that

$$
\left(\mathcal{E}_R(u-\bar{u}), \frac{R-y}{R}\bar{u}\right)_{L^2(M)} = \frac{R-y}{R}\bar{u}(\mathcal{E}_R(u-\bar{u}),1)_{L^2(M)} = 0
$$

since $\mathcal{E}(u-\bar{u})$ has spatial mean value zero, we have

$$
\begin{aligned}
\left\|\overline{\mathcal{E}}_R u\right\|^2_{L^2(\mathcal{C}_R)} &= \|\mathcal{E}_R(u-\bar{u})\|^2_{L^2(\mathcal{C}_R)} + \left\|\frac{R-y}{R}\bar{u}\right\|^2_{L^2(\mathcal{C}_R)} \\
&\le \frac{1}{2\sqrt{\lambda_1}} \|u-\bar{u}\|^2_{L^2(M)} + \int_0^R \int_M \left(\frac{R-y}{R}\right)^2 |\bar{u}|^2 \\
&\le \frac{1}{2\sqrt{\lambda_1}} \|u-\bar{u}\|^2_{L^2(M)} + 4R|M||\bar{u}|^2.
\end{aligned}
$$

□

Lemma 4.2.17. For all $u \in H(M)$,

$$
\left\|\nabla_{\bar{g}}\overline{\mathcal{E}}_R u\right\|^2_{L^2(\mathcal{C}_R)} \le \left(1 + 1/2\sinh^2(\sqrt{\lambda_1}R)\right) |u-\bar{u}|^2_{H(M)} + \frac{|M||\bar{u}|^2}{R}.
$$

where $C(\lambda_1, R) = 1 + 1/2\sinh^2(\sqrt{\lambda_1}R)$.

Proof. Let first $\bar{u} = 0$. We have from Lemma 4.2.15 that

$$\int_M |\nabla_{\bar{g}} v(y)|^2 = 2 \sum_{k=1}^{\infty} |(u, \varphi_k)_{L^2}|^2 \lambda_k \left(\alpha_1(k,R)^2 e^{2\sqrt{\lambda_k} y} + \alpha_2(k,R)^2 e^{-2\sqrt{\lambda_k} y} \right),$$

and integrating over $[\epsilon, R]$, with the aid of the uniform convergence property in Lemma 4.2.15 and the calculation

$$\begin{aligned}\int_\epsilon^R \alpha_1(k,R)^2 e^{2\sqrt{\lambda_k} y} + \alpha_2(k,R)^2 e^{-2\sqrt{\lambda_k} y} &= \frac{1}{2\sqrt{\lambda_k}} \frac{e^{2\sqrt{\lambda_k}(R-\epsilon)} - 1 + 1 - e^{-2\sqrt{\lambda_k}(R-\epsilon)}}{(e^{\sqrt{\lambda_k} R} - e^{-\sqrt{\lambda_k} R})^2} \\ &= \frac{1}{2\sqrt{\lambda_k}} \frac{e^{2\sqrt{\lambda_k}(R-\epsilon)} - e^{-2\sqrt{\lambda_k}(R-\epsilon)}}{e^{2\sqrt{\lambda_k} R} + e^{-2\sqrt{\lambda_k} R} - 2},\end{aligned}$$

we find

$$\int_\epsilon^R \int_M |\nabla_{\bar{g}} v|^2 = \sum_{k=1}^{\infty} |(u, \varphi_k)_{L^2}|^2 \sqrt{\lambda_k} \frac{e^{2\sqrt{\lambda_k}(R-\epsilon)} - e^{-2\sqrt{\lambda_k}(R-\epsilon)}}{e^{2\sqrt{\lambda_k} R} + e^{-2\sqrt{\lambda_k} R} - 2}.$$

Note that

$$\begin{aligned}\frac{e^{2\sqrt{\lambda_k}(R-\epsilon)} - e^{-2\sqrt{\lambda_k}(R-\epsilon)}}{e^{2\sqrt{\lambda_k} R} + e^{-2\sqrt{\lambda_k} R} - 2} &\leq \frac{e^{2\sqrt{\lambda_k} R} + e^{-2\sqrt{\lambda_k} R}}{e^{2\sqrt{\lambda_k} R} + e^{-2\sqrt{\lambda_k} R} - 2} \\ &= 1 + \frac{2}{e^{2\sqrt{\lambda_k} R} + e^{-2\sqrt{\lambda_k} R} - 2} \\ &= 1 + \frac{2}{(2\sinh(\sqrt{\lambda_k} R))^2} \\ &\leq 1 + \frac{1}{2\sinh^2(\sqrt{\lambda_1} R)},\end{aligned}$$

therefore,

$$\int_\epsilon^R \int_M |\nabla_{\bar{g}} v|^2 \leq \left(1 + \frac{1}{2\sinh^2(\sqrt{\lambda_1} R)} \right) \sum_{k=1}^{\infty} \sqrt{\lambda_k} |(u, \varphi_k)_{L^2}|^2.$$

Passing to the limit as $\epsilon \to 0$, using the monotone convergence theorem on the left hand side, we find

$$\int_0^R \int_M |\nabla_{\bar{g}} \mathcal{E}_R u|^2 \leq C(\lambda_1, R) \sum_{k=1}^{\infty} \sqrt{\lambda_k} |(u, \varphi_k)_{L^2}|^2 = C(\lambda_1, R) \, |u|^2_{H(M)}.$$

For the non-zero mean value case, noting that

$$\left(\nabla_{\bar{g}}\mathcal{E}_R(u-\bar{u}), \nabla_{\bar{g}}\left(\frac{R-y}{R}\bar{u}\right)\right)_{L^2(M)} = -\frac{\bar{u}}{R}(\partial_y\mathcal{E}_R(u-\bar{u}), 1)_{L^2(M)} = 0$$

again due to the mean value zero, we have

$$\begin{aligned}
\left\|\nabla_{\bar{g}}\overline{\mathcal{E}}_R u\right\|^2_{L^2(\mathcal{C}_R)} &= \|\nabla_{\bar{g}}\mathcal{E}_R(u-\bar{u})\|^2_{L^2(\mathcal{C}_R)} + \left\|\nabla_{\bar{g}}\left(\frac{R-y}{R}\bar{u}\right)\right\|^2_{L^2(\mathcal{C}_R)} \\
&= \|\nabla_{\bar{g}}\mathcal{E}_R(u-\bar{u})\|^2_{L^2(\mathcal{C}_R)} + \frac{|\bar{u}|^2}{R^2}\|1\|^2_{L^2(\mathcal{C}_R)} \\
&\leq C(\lambda_1, R)\,|u-\bar{u}|^2_{H(M)} + \frac{|M||\bar{u}|^2}{R}.
\end{aligned}$$

$\square$

Remark 4.2.18. Define a form $a_R \colon H(M) \times H(M) \to \mathbb{R}$ by

$$a_R(u,\eta) = \int_{\mathcal{C}_R} \nabla_{\bar{g}}\overline{\mathcal{E}}_R u \nabla_{\bar{g}}\tilde{\eta}$$

where $\tilde{\eta} \in H^1_0(\mathcal{C}_R)$ is an (arbitrary) extension of η; the choice of extension does not matter, since for any two such extensions $\tilde{h}_1$ and $\tilde{h}_2$,

$$\int_{\mathcal{C}_R} \nabla_{\bar{g}}\overline{\mathcal{E}}_R u \nabla_{\bar{g}}\tilde{\tilde{\eta}}_1 - \int_{\mathcal{C}_R} \nabla_{\bar{g}}\overline{\mathcal{E}}_R u \nabla_{\bar{g}}\tilde{\eta}_2 = \int_{\mathcal{C}_R} \nabla_{\bar{g}}\overline{\mathcal{E}}_R u \nabla_{\bar{g}}(\tilde{\eta}_1 - \tilde{\eta}_2) = 0$$

by definition of the weak solution, because $\tilde{\eta}_1 - \tilde{\eta}_2 \in H^1_0(\mathcal{C}_R)$.

4.2.5 Decay and convergence of solutions of the truncated problem

Definition 4.2.19. For any $R > 0$, define the zero extension $\mathscr{Z}_R \colon \{\eta \in H^1(0,R) \mid \eta(R) = 0\} \to H^1(0,\infty)$ by

$$(\mathscr{Z}_R\eta)(y) = \begin{cases} \eta(y) & : \text{if } y \leq R \\ 0 & : \text{otherwise.} \end{cases}$$

This satisfies $\|\mathscr{Z}_R\eta\|_{H^1(0,\infty)} = \|\eta\|_{H^1(0,R)}$.

Clearly, we can also view $\mathscr{Z}_R$ as a map $\mathscr{Z}_R \colon \{\eta \in H^1(\mathcal{C}_R) \mid \eta(x,R) = 0\} \to H^1(\mathcal{C})$ and this satisfies $\|\mathscr{Z}_R\eta\|_{H^1(\mathcal{C})} = \|\eta\|_{H^1(\mathcal{C}_R)}$.

Lemma 4.2.20. For all $u \in H(M)$,

$$\left\|\nabla_{\bar{g}}(\bar{\mathcal{E}}u - \mathscr{Z}_R\overline{\mathcal{E}}_Ru)\right\|^2_{L^2(\mathcal{C})} \le 3e^{-R\sqrt{\lambda_1}}\left|u-\bar{u}\right|^2_{H(M)} + \frac{4}{R}e^{-2R\sqrt{\lambda_1}}\left\|u-\bar{u}\right\|^2_{L^2(M)} + \frac{4|M||\bar{u}|^2}{R}$$

and thus $\mathscr{Z}_R\overline{\mathcal{E}}_Ru \to \bar{\mathcal{E}}u$ in $X(\mathcal{C})$ as $R \to \infty$.

Proof. Let $\eta_R = (\bar{\mathcal{E}}u - \overline{\mathcal{E}}_Ru) - \bar{\mathcal{E}}u(R)\frac{y}{R}$ which satisfies $\eta_R(0) = 0$ and $\eta_R(R) = 0$, consider the difference of the weak formulations of $\overline{\mathcal{E}}_Ru$ tested with η_R and $\overline{\mathcal{E}}u$ tested with $\mathscr{Z}_R\eta_R$:

$$\int_0^R\int_M \nabla_{\bar{g}}\overline{\mathcal{E}}_Ru\nabla_{\bar{g}}\eta_R = 0 \quad \text{and} \quad \int_0^\infty\int_M \nabla_{\bar{g}}\bar{\mathcal{E}}u\nabla_{\bar{g}}\mathscr{Z}_R\eta_R = \int_0^R\int_M \nabla_{\bar{g}}\bar{\mathcal{E}}u\nabla_{\bar{g}}\eta_R = 0$$

which is

$$\int_0^R\int_M |\nabla_{\bar{g}}(\bar{\mathcal{E}}u - \overline{\mathcal{E}}_Ru)|^2 - \nabla_{\bar{g}}(\bar{\mathcal{E}}u - \overline{\mathcal{E}}_Ru)\nabla_{\bar{g}}(\bar{\mathcal{E}}u(R)\frac{y}{R}) = 0,$$

so

$$\begin{aligned}
&\int_0^R\int_M |\nabla_{\bar{g}}(\bar{\mathcal{E}}u - \overline{\mathcal{E}}_Ru)|^2 \\
&\quad\le \int_0^R\int_M |\nabla_{\bar{g}}(\bar{\mathcal{E}}u - \overline{\mathcal{E}}_Ru)||\nabla_{\bar{g}}(\bar{\mathcal{E}}u(R))| + |\partial_y(\bar{\mathcal{E}}u - \overline{\mathcal{E}}_Ru)||\bar{\mathcal{E}}u(R)|\frac{1}{R} \\
&\quad\le \frac{1}{2}\left\|\nabla_{\bar{g}}(\bar{\mathcal{E}}u - \overline{\mathcal{E}}_Ru)\right\|^2_{L^2(\mathcal{C}_R)} + \int_0^R\int_M |\nabla_\Gamma\bar{\mathcal{E}}u(R)|^2 + \frac{1}{R^2}|\bar{\mathcal{E}}u(R)|^2
\end{aligned}$$

where we used $ab \le \frac{a^2}{4} + b^2$. Now, since $\bar{\mathcal{E}}u(R) = \sum_{k\ge1} e^{-R\sqrt{\lambda_k}}(u-\bar{u},\varphi_k)\varphi_k + \bar{u}$,

$$\begin{aligned}
\int_0^R\int_M |\bar{\mathcal{E}}u(R)|^2 &= R\sum_{k\ge1} e^{-2R\sqrt{\lambda_k}}|(u-\bar{u},\varphi_k)|^2 + R|M||\bar{u}|^2 \\
&\le Re^{-2R\sqrt{\lambda_1}}\left\|u-\bar{u}\right\|^2_{L^2(M)} + R|M||\bar{u}|^2
\end{aligned}$$

and

$$\begin{aligned}
\int_0^R\int_M |\nabla_\Gamma\bar{\mathcal{E}}u(R)|^2 &= \sum_{k\ge1} R\lambda_k e^{-2R\sqrt{\lambda_k}}|(u-\bar{u},\varphi_k)|^2 \\
&\le \sum_{k\ge1} \sqrt{\lambda_k}e^{-R\sqrt{\lambda_k}}|(u-\bar{u},\varphi_k)|^2 \qquad (\text{using } xe^{-2x} \le e^{-x}) \\
&\le e^{-R\sqrt{\lambda_1}}\left|u-\bar{u}\right|^2_{H(M)},
\end{aligned}$$

giving

$$\int_0^R \int_M |\nabla_{\bar{g}}(\bar{\mathcal{E}}u - \overline{\mathcal{E}}_R u)|^2 \leq 2e^{-R\sqrt{\lambda_1}} \, |u - \bar{u}|^2_{H(M)} + \frac{2}{R} e^{-2R\sqrt{\lambda_1}} \, \|u - \bar{u}\|^2_{L^2(M)} + \frac{2|M||\bar{u}|^2}{R}.$$

Secondly, note that (using the uniform convergence in Lemma 4.2.5 to integrate term by term)

$$\begin{aligned}
\int_R^\infty \int_M |\nabla_{\bar{g}} \bar{\mathcal{E}} u|^2 &= 2 \int_R^\infty \sum_{k \geq 1} \lambda_k e^{-2y\sqrt{\lambda_k}} |(u - \bar{u}, \varphi_k)|^2 \\
&= \sum_{k \geq 1} \sqrt{\lambda_k} e^{-2R\sqrt{\lambda_k}} |(u - \bar{u}, \varphi_k)|^2 \\
&= e^{-2R\sqrt{\lambda_1}} \, |u - \bar{u}|^2_{H(M)} \, .
\end{aligned}$$

□

Lemma 4.2.21. For all $u \in H(M)$ with $\bar{u} = 0$,

$$\int_0^\infty \int_M |\mathscr{Z}_R \mathcal{E}_R u - \mathcal{E} u|^2 \leq C_P \left(3e^{-R\sqrt{\lambda_1}} \, |u|^2_{H(M)} + \frac{2}{R} e^{-2R\sqrt{\lambda_1}} \, \|u\|^2_{L^2(M)} \right) + \frac{e^{-2R\lambda_1^{1/2}}}{2\lambda_1^{1/2}} \, \|u\|^2_{L^2(M)}$$

(where C_P is the Poincaré constant on M) and thus $\mathscr{Z}_R \mathcal{E}_R u \to \mathcal{E} u$ in $L^2(\mathcal{C})$.

Proof. If $\bar{u} = 0$, then $\overline{\mathcal{E}u(y)} = \overline{\mathcal{E}_R u(y)} = 0$ for all y. Therefore, Poincaré's inequality on M gives for $y > 0$

$$\int_M |\mathcal{E}u(y) - \mathcal{E}_R u(y)|^2 \leq C_P \int_M |\nabla_M \mathcal{E} u(y) - \nabla_M \mathcal{E}_R u(y)|^2$$

for a.a. y. Therefore,

$$\begin{aligned}
\int_0^R \int_M |\mathcal{E}u - \mathcal{E}_R u|^2 &\leq C_P \int_0^R \int_M |\nabla_{\bar{g}} \mathcal{E} u - \nabla_{\bar{g}} \mathcal{E}_R u|^2 \\
&\leq C_P \int_0^\infty \int_M |\nabla_{\bar{g}} \mathcal{E} u - \nabla_{\bar{g}} \mathscr{Z}_R \mathcal{E}_R u|^2 \\
&\leq C_P K(R, u)
\end{aligned}$$

where $K(R,u)$ is from the previous lemma. Over the interval (R,∞), we have

$$\begin{aligned}\int_R^\infty \int_M |\mathscr{Z}_R\mathcal{E}_R u - \mathcal{E}u|^2 &= \int_R^\infty \int_M |\mathcal{E}u|^2\\ &= \int_R^\infty \sum_{k=1}^\infty e^{-2y\lambda_k^{1/2}} |(u,\varphi_k)_{L^2(M)}|^2\\ &= \sum_{k=1}^\infty \frac{e^{-2R\lambda_k^{1/2}}}{2\lambda_k^{1/2}} |(u,\varphi_k)_{L^2(M)}|^2\\ &\quad \text{(by the uniform convergence of Lemma 4.2.5)}\\ &\le \frac{e^{-2R\lambda_1^{1/2}}}{2\lambda_1^{1/2}} \|u\|^2_{L^2(M)}.\end{aligned}$$

□

The next lemma describes *continuous convergence*.

Lemma 4.2.22. If u_R, $u \in H^{1/2}(M)$ with $u_R \to u$ in $L^2(M)$ with $\bar{u}_R = \bar{u} = 0$, then $\mathscr{Z}_R\mathcal{E}_R u_R \to \mathcal{E}u$ in $L^2(\mathcal{C})$.

Proof. We have, writing $\mathscr{Z}_R\mathcal{E}_R u_R - \mathcal{E}u = \mathscr{Z}_R\mathcal{E}_R u_R - \mathscr{Z}_R\mathcal{E}_R u + \mathscr{Z}_R\mathcal{E}_R u - \mathcal{E}u$,

$$\begin{aligned}\|\mathscr{Z}_R\mathcal{E}_R u_R - \mathcal{E}u\|_{L^2(\mathcal{C})} &\le \|\mathscr{Z}_R\mathcal{E}_R(u_R-u)\|_{L^2(\mathcal{C})} + \|\mathscr{Z}_R\mathcal{E}_R u - \mathcal{E}u\|_{L^2(\mathcal{C})}\\ &= \|\mathcal{E}_R(u_R-u)\|_{L^2(\mathcal{C}_R)} + \|\mathscr{Z}_R\mathcal{E}_R u - \mathcal{E}u\|_{L^2(\mathcal{C})}\\ &\le \left(\frac{1}{2\sqrt{\lambda_1}}\|u_R-u\|^2_{L^2(M)}\right)^{1/2} + \|\mathscr{Z}_R\mathcal{E}_R u - \mathcal{E}u\|_{L^2(\mathcal{C})}\\ &\to 0\end{aligned}$$

(we used Lemma 4.2.16 for the inequality) by assumption and Lemma 4.2.21. □

4.3 Function spaces on evolving hypersurfaces and preliminary results

We start with conditions on the prescribed evolution, in addition to (A_λ).

Assumption 4.3.1. For each $t \in [0,T]$, let $\Gamma(t) \subset \mathbb{R}^{d+1}$ be a compact (i.e., no boundary) d-dimensional smooth hypersurface, and assume the existence of a flow $\Phi\colon [0,T]\times\mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ such that for all $t \in [0,T]$, with $\Gamma_0 := \Gamma(0)$, the map $\Phi_t^0(\cdot) := \Phi(t,\cdot)\colon \Gamma_0 \to \Gamma(t)$ is a C^3-diffeomorphism that satisfies $\frac{d}{dt}\Phi_t^0(\cdot) = \mathbf{w}(t,\Phi_t^0(\cdot))$ and $\Phi_0^0(\cdot) = \mathrm{Id}(\cdot)$ for a given C^2 velocity field $\mathbf{w}\colon [0,T]\times\mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$, which we assume satisfies the uniform bound $|\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)| \leq C$ for all $t \in [0,T]$. A C^2 normal vector field on the hypersurfaces is denoted by $\nu^\Gamma \colon [0,T] \times \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$.

It follows that the Jacobian $J_t^0 := \det \mathbf{D}\Phi_t^0$ is C^2 and is uniformly bounded away from zero and infinity. We denote by $\Phi_0^t \colon \Gamma(t) \to \Gamma_0$ the inverse of Φ_t^0 and define $|\Gamma| := \max_{t\in[0,T]} |\Gamma(t)|$.

Remark 4.3.2. In fact, all functional analytic results in this section not involving the harmonic extension maps are true for $\Gamma(t)$ of class C^3. The assumption (A_λ) is satisfied if for example each $\Gamma(t)$ has non-negative Ricci curvature, or if the Ricci curvature of $\Gamma(t)$ is greater than $\rho(t) < 0$, where $-\rho(t) \leq \rho$ holds for all $t \in [0,T]$ with ρ a constant. See Theorem 4.6.1 in [76] and the discussion afterwards. Also, instead of assuming (A_λ), one could study the possible continuity of $t \mapsto \lambda_1(t)$ through the theory of perturbations of linear operators [78].

4.3.1 Function spaces

In order to define the spaces L^p_Y mentioned in the introduction, we need simply to check a few assumptions.

Spaces on the surface Γ

For $u \in L^2(\Gamma_0)$, define $(\phi_t u)(x) := (\phi_{\Gamma,t} u)(x) := u(\Phi_0^t(x))$. Fortunately, we already checked that the spaces $L^p_{L^q}$ and $L^2_{W^{1/2,2}}$ are well-defined in §3.2.1 and §2.5.4 respectively; in the latter section the evolving Sobolev–Bochner space

$$\mathbb{W}(W^{1/2,2}, W^{-1/2,2}) = \{u \in L^2_{W^{1/2,2}} \mid \dot{u} \in L^2_{W^{-1/2,2}}\}$$

was shown (see Theorem 2.5.6) to be well-defined and isomorphic (via $\phi_{\Gamma,-(\cdot)}$) with an equivalence of norms to

$$\mathcal{W}(W^{1/2,2}, W^{-1/2,2}) := \{\tilde{u} \in L^2(0,T; W^{1/2,2}(\Gamma_0)) \mid \tilde{u}' \in L^2(0,T; W^{-1/2,2}(\Gamma_0))\}.$$

Lemma 4.3.3. The space $\mathbb{W}(W^{1/2,2}, W^{-1/2,2})$ is compactly embedded in $L^2_{L^2}$.

Proof. That the embedding is continuous is obvious. Let w_n be a bounded sequence in $\mathbb{W}(W^{1/2,2}, W^{-1/2,2})$. Then $\phi_{-(\cdot)} w_n$ is bounded in $W(W^{1/2,2}, W^{-1/2,2})$ and by $\mathcal{W}(W^{1/2,2}, W^{-1/2,2}) \overset{c}{\hookrightarrow} L^2(0,T; L^2(\Gamma_0))$ (by Aubin–Lions), there is a subsequence $\phi_{-(\cdot)} w_{n_k} \to \tilde{w}$ that converges in $L^2(0,T; L^2(\Gamma_0))$. Hence $w_{n_k} \to \phi_{(\cdot)} w$ in $L^2_{L^2}$. $\square$

Spaces on the cylinders $\mathcal{C}$ and $\mathcal{C}_R$

Given $u \in L^2(\mathcal{C}_0)$, define $(\phi_{\mathcal{C},t}u)(x,y) := u(\Phi_0^t(x),y)$. We have

$$\int_0^\infty \int_{\Gamma(t)} |\phi_{\mathcal{C},t}u|^2 = \int_0^\infty \int_{\Gamma(t)} |u(\Phi_0^t(x),y)|^2 = \int_0^\infty \int_{\Gamma_0} |u(z,y)|^2 J_t^0 \leq C_J \|u\|^2_{L^2(\mathcal{C})},$$

so $\phi_{\mathcal{C},t} \colon L^2(\mathcal{C}_0) \to L^2(\mathcal{C}(t))$. The inverse mapping is $\phi_{\mathcal{C},-t} \colon L^2(\mathcal{C}(t)) \to L^2(\mathcal{C}_0)$ given by $(\phi_{\mathcal{C},-t}w)(x,y) = w(\Phi_t^0(x),y)$ and these maps are linear homeomorphisms. Also, we see that if $u \in H^1(\mathcal{C}_0)$,

$$\begin{aligned}
\int_0^\infty \int_{\Gamma(t)} |\nabla_{\bar{g}}\phi_{\mathcal{C},t}u|^2 &= \int_0^\infty \int_{\Gamma(t)} |\nabla_{\bar{g}}u(\Phi_0^t(x),y)|^2 \\
&= \int_0^\infty \int_{\Gamma(t)} |(\mathbf{D}\Phi_0^t)^\intercal(x)(\nabla_{\Gamma_0}u(y) \circ \Phi_0^t(x)) + \partial_y u(\Phi_0^t(x),y)|^2 \\
&\leq 2\int_0^\infty \int_{\Gamma_0} |(\mathbf{D}\Phi_0^t)^\intercal \circ \Phi_t^0(z)(\nabla_{\Gamma_0}u(z,y))|^2 J_t^0 + |\partial_y u(z,y)|^2 J_t^0 \\
&\leq C_1 \int_0^\infty \int_{\Gamma_0} |\nabla_{\Gamma_0}u(z,y)|^2 + |\partial_y u(z,y)|^2 \\
&= C_1 |u|^2_{H^1(\mathcal{C}_0)},
\end{aligned}$$

which shows that

$$|\phi_{\mathcal{C},t}u|_{H^1(\mathcal{C}(t))} \leq C_2 |u|_{H^1(\mathcal{C}_0)}. \tag{4.16}$$

Overall, we have shown that $\|\phi_{\mathcal{C},t}u\|_{H^1(\mathcal{C}(t))} \leq C_3 \|u\|_{H^1(\mathcal{C}_0)}$ for all t and $u \in H^1(\mathcal{C}_0)$ and that $\phi_{\mathcal{C},t} \colon H^1(\mathcal{C}_0) \to H^1(\mathcal{C}(t))$ is also well-defined. Finally, we have

$$t \mapsto \|\phi_{\mathcal{C},t}u\|^2_{H^1(\mathcal{C}(t))} = \int_{\mathcal{C}_0} |u(z,y)|^2 J_t^0 + |(\mathbf{D}\Phi_0^t)^\intercal \circ \Phi_t^0(z)(\nabla_{\Gamma_0}u(z,y)) + \partial_y u(z,y)|^2 J_t^0$$

is continuous. This allows us to define the spaces $L^2_{H^1(\mathcal{C})}$ and $L^2_{L^2(\mathcal{C})}$ (just ignore the gradient term). Clearly the same argument allows us to define $L^2_{L^2(\mathcal{C}_R)}$, $L^2_{H^1(\mathcal{C}_R)}$, and $L^2_{H^1_0(\mathcal{C}_R)}$ using a map $\phi_{\mathcal{C}_R,t}$ defined in the same way.

Definition 4.3.4. We denote by $\overline{\mathcal{E}}_t$ and $\overline{\mathcal{E}}_{R,t}$ the maps $\overline{\mathcal{E}}$ and $\overline{\mathcal{E}}_R$ defined in Theorems 4.2.4 and 4.2.13 respectively with the manifold M chosen to be $\Gamma(t)$ (and likewise without the bars). Similarly, we denote by $\overline{\mathcal{T}}_t$, $\mathcal{T}_{R,t,y=0}$ and $\mathcal{T}_{R,t,y=R}$ the trace maps $\overline{\mathcal{T}}$, $\mathcal{T}_{R,y=0}$ and $\mathcal{T}_{R,y=R}$ defined in Lemma 4.2.10 and in §4.2.1 respectively with the choice $M = \Gamma(t)$.

Lemma 4.3.5 (Commutativity of the trace and pushforward maps)**.** The following identity holds:

$$\mathcal{T}_t(\phi_{\mathcal{C},t}v) = \phi_{\Gamma,t}(\mathcal{T}_0 v) \quad \text{for all } v \in H^1(\mathcal{C}_0).$$

Proof. We have $\phi_{\mathcal{C},t}v = v \circ \Phi_0^t \in H^1(\mathcal{C}(t))$ and so $\mathcal{T}_t\phi_{\mathcal{C},t}v = v(\Phi_0^t(\cdot), 0)$, whilst on the other hand, $\phi_{\Gamma,t}\mathcal{T}_0 v = v(\cdot, 0) \circ \Phi_0^t(\cdot) = v(\Phi_0^t(\cdot), 0)$. □

Lemma 4.3.5 implies that if $v \in H^1(\mathcal{C}(t))$, then, using the boundedness of $\phi_{\Gamma,t}$,

$$\|\mathcal{T}_t v\|_{W^{1/2,2}(\Gamma(t))} \leq C_1 \|\mathcal{T}_0\phi_{\mathcal{C},-t}v\|_{W^{1/2,2}(\Gamma_0)} \leq C_2 \|\phi_{\mathcal{C},-t}v\|_{H^1(\mathcal{C}_0)} \leq C_3 \|v\|_{H^1(\mathcal{C}(t))} \tag{4.17}$$

because of the trace theorem and the equivalence of norms between $H^{1/2}(\Gamma_0)$ and $W^{1/2,2}(\Gamma_0)$. This shows that $\mathcal{T}_t\colon H^1(\mathcal{C}(t)) \to W^{1/2,2}(\Gamma(t))$ is bounded independently of t. By the same argument, the maps $\mathcal{T}_{R,t,y=0}, \mathcal{T}_{R,t,y=R}\colon H^1(\mathcal{C}_R(t)) \to W^{1/2,2}(\Gamma(t))$ are also bounded uniformly in t. Now, by Lemma 4.3.5 and (4.16), we have for $v \in H^1(\mathcal{C}_0)$

$$\|\phi_{\mathcal{C},t}v\|^2_{X(\mathcal{C}(t))} = |\phi_{\mathcal{C},t}v|^2_{H^1(\mathcal{C}(t))} + \|\mathcal{T}_t\phi_{\mathcal{C},t}v\|^2_{L^2(\Gamma(t))} \leq C\|v\|^2_{X(\mathcal{C}_0)},$$

which shows that $\phi_{\mathcal{C},t}\colon H^1(\mathcal{C}_0) \to X(\mathcal{C}(t))$ has a useful boundedness property which, by the Bounded Linear Transformation (BLT) theorem, allows us to extend $\phi_{\mathcal{C},t}$ to a bounded linear map $\bar{\phi}_{\mathcal{C},t}\colon X(\mathcal{C}_0) \to X(\mathcal{C}(t))$ defined as

$$\bar{\phi}_{\mathcal{C},t}x_0 := \lim_{n\to\infty} \phi_{\mathcal{C},t}v_n \quad \text{in } X(\mathcal{C}(t)) \text{ for } v_n \in H^1(\mathcal{C}_0) \text{ with } v_n \to x_0 \text{ in } X(\mathcal{C}_0).$$

We also have the measurability of $t \mapsto \|\bar{\phi}_{\mathcal{C},t}x_0\|_{X(\mathcal{C}(t))} = \lim_{n\to\infty} \|\phi_{\mathcal{C},t}v_n\|_{X(\mathcal{C}(t))}$. Thus $L^2_{X(\mathcal{C})}$ is also well-defined. Similar arguments can be made for the inverse operator of $\bar{\phi}_{\mathcal{C},t}$, denoted $\bar{\phi}_{\mathcal{C},-t}\colon X(\mathcal{C}(t)) \to X(\mathcal{C}_0)$.

Some bounds

When we work with a time-dependent manifold $M = \Gamma(t)$, we would like the constants in the gradient bounds (4.12) and in Lemma 4.2.17 to be independent of time. The space $H^{1/2}(\Gamma(t))$ is equivalent to $W^{1/2,2}(\Gamma(t))$ with an equivalence of norms, as we mentioned in the introduction. However, the constants in the equivalence of norms result will depend on t and we have no information as to in what way the dependence is. This means that one has to be careful whenever one uses estimates from §4.2 involving the $H^{1/2}(\Gamma(t))$ or $H(\Gamma(t))$ seminorm in the evolving set-up. For this reason, we need the bounds in the next two lemmas.

Lemma 4.3.6. For all t and all $u \in H^{1/2}(\Gamma(t))$,

$$\left\|\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t u\right\|_{L^2(\mathcal{C}(t))} \leq C \left\|u - \bar{u}\right\|_{W^{1/2,2}(\Gamma(t))}$$

where C is independent of t.

Proof. First suppose that $\bar{u} = 0$ and set $U := \phi_{\mathcal{C},t}\mathcal{R}_0\phi_{\Gamma,-t}u \in H^1(\mathcal{C}(t))$ where $\mathcal{R}_0 \colon H^{1/2}(\Gamma_0) \to H^1(\mathcal{C}_0)$ is the right continuous extension operator which is the partial inverse of the trace operator $\mathcal{T}_0$. Note that, using the equivalence of norms of $H^{1/2}(\Gamma_0)$ and $W^{1/2,2}(\Gamma_0)$,

$$\|U\|_{H^1(\mathcal{C}(t))} \leq C_0 \left\|\mathcal{R}_0\phi_{\Gamma,-t}u\right\|_{H^1(\mathcal{C}_0)} \leq C_1 \left\|\phi_{\Gamma,-t}u\right\|_{H^{1/2}(\Gamma_0)} \leq C_3 \|u\|_{W^{1/2,2}(\Gamma(t))}.$$

Also, we have $\mathcal{T}_t U = \mathcal{T}_t\phi_{\mathcal{C},t}\mathcal{R}_0\phi_{\Gamma,-t}u = \phi_{\Gamma,t}\mathcal{T}_0\mathcal{R}_0\phi_{\Gamma,-t}u = u$ by Lemma 4.3.5. So the function $\eta = \mathcal{E}_t u - U \in H^1(\mathcal{C}(t))$ satisfies $\mathcal{T}_t\eta = u - u = 0$ and is an admissible test function in the weak formulation:

$$\begin{aligned}
\int_{\mathcal{C}(t)} |\nabla_{\bar{g}(t)}\mathcal{E}_t u|^2 &= \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)}\mathcal{E}_t u \nabla_{\bar{g}(t)} U \\
&\leq \frac{1}{2}\int_{\mathcal{C}(t)} |\nabla_{\bar{g}(t)}\mathcal{E}_t u|^2 + \frac{1}{2}\int_{\mathcal{C}(t)} |\nabla_{\bar{g}(t)} U|^2 \\
&\leq \frac{1}{2}\int_{\mathcal{C}(t)} |\nabla_{\bar{g}(t)}\mathcal{E}_t u|^2 + \frac{C}{2}\|u\|^2_{W^{1/2,2}(\Gamma)}.
\end{aligned}$$

For general u, the identity follows from $\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t u = \nabla_{\bar{g}(t)}\mathcal{E}_t(u - \bar{u})$. $\square$

Lemma 4.3.7 (cf. Lemma 4.3.6)**.** For all t and all $u \in H^{1/2}(\Gamma(t))$,

$$\left\|\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}u\right\|^2_{L^2(\mathcal{C}_R(t))} \leq C_1 \|u - \bar{u}\|^2_{W^{1/2,2}(\Gamma(t)} + \frac{C_2}{R^2}\|u - \bar{u}\|^2_{L^2(\Gamma(t))} + \frac{2|\bar{u}|^2}{R}|\Gamma|$$

where C_1 and C_2 are independent of t.

Proof. Suppose $\bar{u} = 0$ and let $\eta = \mathcal{E}_{R,t}u - \frac{R-y}{R}\mathcal{E}_t u \in H^1(\mathcal{C}(t))$ which satisfies $\mathcal{T}_{R,t,y=0}\eta = u - u = 0$ and $\mathcal{T}_{R,t,y=R}\eta = 0$ and thus is a valid test function in the weak formulation:

$$\begin{aligned}
\int_0^R \int_{\Gamma(t)} |\nabla_{\bar{g}(t)}\mathcal{E}_{R,t}u|^2 &\leq \int_0^R \int_{\Gamma(t)} |\nabla_{\bar{g}(t)}\mathcal{E}_{R,t}u| \left|\nabla_{\bar{g}(t)}\left(\frac{R-y}{R}\mathcal{E}_t u\right)\right| \\
&\leq \frac{1}{2}\int_0^R \int_{\Gamma(t)} |\nabla_{\bar{g}(t)}\mathcal{E}_{R,t}u|^2 + \left|\nabla_{\bar{g}(t)}\left(\frac{R-y}{R}\mathcal{E}_t u\right)\right|^2,
\end{aligned}$$

which gives

$$\begin{aligned}
\int_0^R \int_{\Gamma(t)} |\nabla_{\bar{g}(t)} \mathcal{E}_{R,t} u|^2 &\leq \int_0^R \int_{\Gamma(t)} \left| \frac{R-y}{R} \nabla_{\bar{g}(t)} \mathcal{E}_t u - \frac{1}{R} \mathcal{E}_t u \right|^2 \\
&\leq 2 \int_0^R \int_{\Gamma(t)} 4 |\nabla_{\bar{g}(t)} \mathcal{E}_t u|^2 + \frac{1}{R^2} |\mathcal{E}_t u|^2 \\
&\leq C_1 \|u\|^2_{W^{1/2,2}} + \frac{C_2}{R^2} \|u\|^2_{L^2(\Gamma(t))}
\end{aligned}$$

where we bounded the integral with R by ∞ (this is why we need $\bar{u} = 0$) and used Lemma 4.3.6 and (4.11) in conjunction with (A_λ). For the $\bar{u} \neq 0$ case,

$$\begin{aligned}
\int_0^R \int_{\Gamma(t)} |\nabla_{\bar{g}(t)} \overline{\mathcal{E}}_{R,t} u|^2 &= \int_0^R \int_{\Gamma(t)} |\nabla_{\bar{g}(t)} \mathcal{E}_{R,t}(u - \bar{u}) + \nabla_{\bar{g}(t)} \frac{R-y}{R} \bar{u}|^2 \\
&= \int_0^R \int_{\Gamma(t)} |\nabla_{\bar{g}(t)} \mathcal{E}_{R,t}(u - \bar{u}) - \frac{\bar{u}}{R}|^2 \\
&\leq C_1 \|u - \bar{u}\|^2_{W^{1/2,2}(\Gamma(t))} + \frac{C_2}{R^2} \|u - \bar{u}\|^2_{L^2(\Gamma(t))} + \frac{2|\bar{u}|^2}{R} |\Gamma|.
\end{aligned}$$

□

Superposition trace maps

Lemma 4.3.8. There exists a bounded linear trace operator $\mathbb{T}\colon L^2_{H^1(\mathcal{C})} \to L^2_{W^{1/2,2}}$ satisfying $(\mathbb{T}v)(t) = \mathcal{T}_t v(t)$ for almost every t.

Proof. Define $(\mathbb{T}v)(t) = \mathcal{T}_t v(t)$ with the trace map $\mathcal{T}_t\colon H^1(\mathcal{C}(t)) \to W^{1/2,2}(\Gamma(t))$ as before. We have by Lemma 4.3.5 that $(\mathbb{T}v)(t) = \phi_{\Gamma,t}(\mathcal{T}_0(\phi_{\mathcal{C},-t} v(t)))$ so $t \mapsto \phi_{\Gamma,-t}(\mathbb{T}v)(t)$ is measurable, and

$$\|\mathbb{T}v\|^2_{L^2_{W^{1/2,2}}} = \int_0^T \|\mathcal{T}_t v(t)\|^2_{W^{1/2,2}(\Gamma(t))} \leq C \int_0^T \|v(t)\|^2_{H^1(C(t))} = C \|v\|^2_{L^2_{H^1(\mathcal{C})}}$$

by (4.17). □

Lemma 4.3.9. There exists a bounded linear trace operator $\overline{\mathbb{T}}\colon L^2_{X(\mathcal{C})} \to L^2_{W^{1/2,2}}$ satisfying $(\overline{\mathbb{T}}v)(t) = \overline{\mathcal{T}}_t v(t)$ for almost every t.

Proof. Let us first show that $\overline{\mathcal{T}}_t v = \phi_{\Gamma,t} \overline{\mathcal{T}}_0 \bar{\phi}_{\mathcal{C},-t} v$ for all $v \in X(\mathcal{C}(t))$. Given $v \in X(\mathcal{C}(t))$, let $v_n \in H^1(\mathcal{C}(t))$ with $v_n \to v$ in $X(\mathcal{C}(t))$. The result of Lemma 4.3.5 is that

$$\overline{\mathcal{T}}_t v_n = \phi_{\Gamma,t} \overline{\mathcal{T}}_0 \bar{\phi}_{\mathcal{C},-t} v_n \tag{4.18}$$

holds. It follows that $\bar{\phi}_{\mathcal{C},-t}v_n \to \bar{\phi}_{\mathcal{C},-t}v$ in $X(\mathcal{C}_0)$ and thus $\overline{\mathcal{T}}_0\bar{\phi}_{\mathcal{C},-t}v_n \to \overline{\mathcal{T}}_0\bar{\phi}_{\mathcal{C},-t}v$ in $H^{1/2}(\Gamma_0)$. By continuity, $\phi_{\Gamma,t}\overline{\mathcal{T}}_0\bar{\phi}_{\mathcal{C},-t}v_n \to \phi_{\Gamma,t}\overline{\mathcal{T}}_0\bar{\phi}_{\mathcal{C},-t}v$ in $H^{1/2}(\Gamma(t))$. The left hand side of (4.18) converges to $\overline{\mathcal{T}}_tv$ by similar arguments, so the identity we wished to show holds. Then for $v \in L^2_{X(\mathcal{C})}$, we have

$$(\overline{\mathbb{T}}v)(t) = \overline{\mathcal{T}}_tv(t) = \phi_{\Gamma,t}\overline{\mathcal{T}}_0\bar{\phi}_{\mathcal{C},-t}v(t)$$

which gives measurability in time, and we have the bound

$$\begin{aligned}
\left\|(\overline{\mathbb{T}}v)(t)\right\|_{W^{1/2,2}(\Gamma(t))} &\leq C_1\left\|\overline{\mathcal{T}}_0\bar{\phi}_{\mathcal{C},-t}v(t)\right\|_{W^{1/2,2}(\Gamma_0)}\\
&\leq C_3\left\|\overline{\mathcal{T}}_0\bar{\phi}_{\mathcal{C},-t}v(t)\right\|_{H(\Gamma_0)}\\
&\leq C_3\left\|\bar{\phi}_{\mathcal{C},-t}v(t)\right\|_{X(\mathcal{C}_0)} && \text{(by Lemma 4.2.10)}\\
&\leq C_4\left\|v(t)\right\|_{X(\mathcal{C}(t))}
\end{aligned}$$

which proves that $\overline{\mathbb{T}}\colon L^2_{X(\mathcal{C})} \to L^2_{W^{1/2,2}}$ is well-defined as a bounded linear operator. $\square$

Lemma 4.3.10. There exist bounded linear trace maps $\mathbb{T}_{R,y=0}, \mathbb{T}_{R,y=R}\colon L^2_{H^1(C_R)} \to L^2_{W^{1/2,2}}$ satisfying $(\mathbb{T}_{R,y=0}v)(t) = \mathcal{T}_{R,t,y=0}v(t)$ and $(\mathbb{T}_{R,y=R}v)(t) = \mathcal{T}_{R,t,y=R}v(t)$ for almost every t.

Proof. Like in Lemma 4.3.5, we can prove that $\mathcal{T}_{R,t,y=0} \circ \phi_{\mathcal{C}_R,t} = \phi_{\Gamma,t} \circ \mathcal{T}_{R,0,y=0}$ and hence that $\mathcal{T}_{R,t,y=0}\colon H^1(\mathcal{C}_R(t)) \to H^{1/2}(\Gamma(t))$ is bounded independently of t, and likewise for $\mathcal{T}_{R,t,y=R}$. Such a result also allows us to define the the superposition maps

$$(\mathbb{T}_{R,y=0}v)(t) := \mathcal{T}_{R,t,y=0}v(t) \qquad \text{and} \qquad (\mathbb{T}_{R,y=R}v)(t) := \mathcal{T}_{R,t,y=R}v(t)$$

just like in Lemma 4.3.8. $\square$

4.3.2 Integration by parts

We will need the following integration by parts results. The first lemma is comparable to a result in [67] and [47, Lemma 7.1].

Lemma 4.3.11. Let $f\colon \mathbb{R} \to \mathbb{R}$ be C^1 and Lipschitz with $f(0) = 0$. Define $F(s) = \int_0^s f(r)\,\mathrm{d}r$. Then for all $u \in \mathbb{W}(W^{1/2,2}, W^{-1/2,2})$, the following formula holds:

$$\int_0^T \langle \dot{u}(t), f(u(t))\rangle = \int_{\Gamma(T)} F(u(T)) - \int_{\Gamma_0} F(u_0) - \int_0^T \int_{\Gamma(t)} F(u(t))\nabla_\Gamma \cdot \mathbf{w}.$$

Proof. Let $u_n \in \mathbb{W}(W^{1/2,2}, L^2) \cap L^\infty_{L^\infty}$ be such that $u_n \to u$ in $\mathbb{W}(W^{1/2,2}, W^{-1/2,2})$. Such a sequence exists because $C^1([0,T] \times \Gamma_0)$ is dense in $\mathcal{W}(W^{1/2,2}, W^{-1/2,2})$ (eg. see Proposition 23.23 in [129] and use density of $C^1(\Gamma_0)$ in $W^{1/2,2}(\Gamma_0)$ [43, Proposition 3.40]). Then $F(u_n) \in \mathbb{W}(W^{1/2,2}, L^2)$. To see this, note that

$$|F(s) - F(t)| \le \int_t^s |f(r)| \le |s-t| \sup_{r \in (s,t)} |f(r)| \le |s-t| \left\|f'\right\|_\infty \max(|s|, |t|),$$

so for almost all t,

$$|F(u_n(t,x))| \le \left\|f'\right\|_\infty |u_n(t,x)|^2 \le \left\|f'\right\|_\infty \|u_n(t)\|^2_{L^\infty(\Gamma(t))}$$

almost everywhere. This gives $\int_{\Gamma(t)} |F(u_n(t))|^2 \le |\Gamma| \|u_n(t)\|^4_{L^\infty(\Gamma(t))} \|f'\|^2_\infty$ and since the right hand side is bounded for a.e. t by $|\Gamma| \|u_n\|^4_{L^\infty_{L^\infty}} \|f'\|^2_\infty$, we have $F(u_n) \in L^2_{L^2}$. We also see that

$$\begin{aligned} |F(u_n(t,x)) - F(u_n(t,y))| &\le \left\|f'\right\|_\infty |u_n(t,x) - u_n(t,y)| \max(|u_n(t,x)|, |u_n(t,y)|) \\ &\le \|u_n(t)\|_{L^\infty(\Gamma(t))} \left\|f'\right\|_\infty |u_n(t,x) - u_n(t,y)| \end{aligned}$$

which shows that $F(u_n) \in L^2_{W^{1/2,2}}$. Likewise, $\partial^\bullet(F(u_n)) = f(u_n)\dot{u}_n \in L^2_{L^2}$. This means the transport theorem is valid and we have

$$\int_0^T \langle \dot{u}_n(t), f(u_n(t)) \rangle = \int_{\Gamma(T)} F(u_n(T)) - \int_{\Gamma_0} F(u_n(0)) - \int_0^T \int_{\Gamma(t)} F(u_n(t)) \nabla_\Gamma \cdot \mathbf{w}. \tag{4.19}$$

We must pass to the limit in n. For a.e. t, we have for a subsequence, $u_n(t) \to u(t)$ in $W^{1/2,2}(\Gamma(t))$, so by Lemma 4.5.3, $\|f(u_n(t)) - f(u(t))\|^2_{W^{1/2,2}(\Gamma(t))} \to 0$ and

$$\|f(u_n(t)) - f(u(t))\|^2_{W^{1/2,2}(\Gamma(t))} \le 2 \left\|f'\right\|^2_\infty \left(\|u_n(t)\|^2_{W^{1/2,2}(\Gamma(t))} + \|u(t)\|^2_{W^{1/2,2}(\Gamma(t))} \right).$$

The right hand side converges to $4 \|f'\|^2_\infty \|u(t)\|^2_{W^{1/2,2}(\Gamma(t))}$ whilst the integral of the right hand side converges to $4 \|f'\|^2_\infty \int_0^T \|u(t)\|^2_{W^{1/2,2}(\Gamma(t))}$ since $u_n \to u$ in $L^2_{W^{1/2,2}}$. Then the generalised dominated convergence theorem (DCT) gives $f(u_n) \to f(u)$

in $L^2_{W^{1/2,2}}$. We see that

$$\begin{aligned}
&\left|\int_{\Gamma(T)} F(u_n(T)) - F(u(T))\right| \\
&\quad \leq \left\|f'\right\|_\infty \int_{\Gamma(T)} |u_n(T) - u(T)| \max(u_n(T), u(T)) \\
&\quad \leq \left\|f'\right\|_\infty \|u_n(T) - u(T)\|_{L^2(\Gamma(T))} (\|u_n(T)\|_{L^2(\Gamma(T))} + \|u(T)\|_{L^2(\Gamma(T))}) \\
&\quad \to 0
\end{aligned}$$

since $u_n(T) \to u(T)$ in $L^2(\Gamma(T))$. This argument deals with the first two terms on the right hand side of (4.19). For the last term,

$$\begin{aligned}
&\left|\int_0^T \int_{\Gamma(t)} F(u_n(t))\nabla_\Gamma \cdot \mathbf{w} - \int_0^T \int_{\Gamma(t)} F(u(t))\nabla_\Gamma \cdot \mathbf{w}\right| \\
&\quad \leq \|\nabla_\Gamma \cdot \mathbf{w}\|_{L^\infty_{L^\infty}} \left\|f'\right\|_\infty \int_0^T \int_{\Gamma(t)} |u_n(t) - u(t)| \max(|u_n(t)|, |u(t)|)
\end{aligned}$$

and we use the same argument as before. □

Lemma 4.3.12. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and piecewise C^1 with $f(0) = 0$ and $f' = 0$ outside a compact set $K \subset\subset \mathbb{R}$. Define $F(s) = \int_0^s f(r)\ \mathrm{d}r$. Then for all $u \in \mathbb{W}(W^{1/2,2}, W^{-1/2,2})$, the formula

$$\int_0^T \langle \dot{u}(t), f(u(t))\rangle = \int_{\Gamma(T)} F(u(T)) - \int_{\Gamma_0} F(u(0)) - \int_0^T \int_{\Gamma(t)} F(u_n(t))\nabla_\Gamma \cdot \mathbf{w}$$

holds.

Remark 4.3.13. Note that $f(u)$ is an element of $L^2_{W^{1/2,2}}$ because f' is bounded a.e. and as f is absolutely continuous it follows that f is Lipschitz.

Proof. Given $u \in \mathbb{W}(W^{1/2,2}, W^{-1/2,2})$, by density, there exist $u_n \in \mathbb{W}(W^{1/2,2}, L^2)$ with $u_n \to u$ in $\mathbb{W}(W^{1/2,2}, W^{-1/2,2})$. We claim now that $F(u_n) \in \mathbb{W}(W^{1/2,2}, L^2)$. To see this, consider

$$\|F(u_n)\|^2_{L^2_{L^2}} = \int_0^T \int_{\Gamma(t)} |F(u_n(t,x))|^2 \leq \mathrm{Lip}(F) \int_0^T \int_{\Gamma(t)} |u_n(t,x)|^2$$

because F is C^1 with $F' = f$ bounded, hence F is Lipschitz. Also, $\partial^\bullet(F(u_n)) = F'(u_n)\dot{u}_n$ implies that $\partial^\bullet(F(u_n)) \in L^2_{L^2}$. The $W^{1/2,2}$ seminorm is also finite because again F is Lipschitz. So $F(u_n) \in \mathbb{W}(W^{1/2,2}, L^2)$. So then we can use the standard integration by parts formula to obtain

$$\int_{\Gamma(T)} F(u_n(T)) - \int_{\Gamma_0} F(u_n(0)) = \int_0^T \langle \dot{u}_n(t), f(u_n(t))\rangle + \int_0^T \int_{\Gamma(t)} F(u_n(t))\nabla_\Gamma \cdot \mathbf{w}. \tag{4.20}$$

We have that $u_n \to u$ in $C^0_{L^1}$ by $\mathbb{W}(W^{1/2,2}, W^{-1/2,2}) \hookrightarrow C^0_{L^2} \hookrightarrow C^0_{L^1}$. This implies that $F(u_n) \to F(u)$ in $C^0_{L^1}$ because

$$\max_{t\in[0,T]} \|F(u_n(t)) - F(u(t))\|_{L^1(\Gamma(t))} \leq \mathrm{Lip}(F) \max_{t\in[0,T]} \|u_n(t) - u(t)\|_{L^1(\Gamma(t))} \to 0.$$

This takes care of the terms on the left hand side of (4.20). Also, $F(u_n) \to F(u)$ in $L^1_{L^1}$ because as mentioned $F(u_n) \to F(u)$ in $C^0_{L^1}$. This is enough since $v \mapsto \int_0^T \int_{\Gamma(t)} v\nabla_\Gamma \cdot \mathbf{w}$ defines a bounded linear functional in $L^1_{L^1}$.

To finish, we need $f(u_n) \to f(u)$ in $L^2_{W^{1/2,2}}$ but this holds due to the reasoning in the proof of Lemma 4.3.11 because f is Lipschitz. □

4.3.3 Truncations

Let Γ be a smooth hypersurface. Define the truncation $T_k \colon \mathbb{R} \to \mathbb{R}$ at height k:

$$T_k(x) = \max\{x, -k\} + \min\{x, k\} - x = \begin{cases} k\,\mathrm{sign}(x) & : |x| \geq k \\ x & : |x| < k. \end{cases}$$

Note that $T_k \colon L^2(\Gamma) \to L^2(\Gamma)$ and $T_k \colon H^1(\Gamma) \to H^1(\Gamma)$ are bounded continuous maps. We have $|\max(x,0) - \max(y,0)| \leq |x-y|$ and $\max(x,-k) = \max(x+k,0) - k$ which implies that $T_k \colon W^{1/2,2}(\Gamma) \to W^{1/2,2}(\Gamma)$ is bounded. Furthermore, the chain rule for weakly differentiable functions u implies that

$$\frac{d}{dz}(T_k u(z)) = \chi_{\{|u(z)|<k\}} \frac{d}{dz} u(z)$$

for almost every z. See [32, Lemma 2.89]) and the discussion after Theorem 4.3.6 in [31] for these facts on a domain Ω.

Now we discuss truncations over cylinders. Suppose $f \in C^1(\mathbb{R})$ with f' bounded and $f(0) = 0$. The chain rule $\nabla_{\bar{g}} f(v) = f'(v)\nabla_{\bar{g}} v$ for $v \in H^1(\mathcal{C})$ can be proved by the standard argument: approximate v by $v_n \in \mathcal{D}([0,\infty); \mathcal{D}(\Gamma))$, prove the identity for v_n and pass to the limit using continuity of f' and the DCT. This then allows us to show

$$\nabla_{\bar{g}} v^+ = \chi_{\{v\geq 0\}} \nabla_{\bar{g}} v$$

(almost everywhere) by approximating $r \mapsto (r)^+$ by C^1 functions with bounded derivatives, the chain rule and then the passage to the limit in the approximations (see [74, Lemma 1.19]). This will imply that if v and w are in $H^1(\mathcal{C})$, then $\max(v,w) \in H^1(\mathcal{C})$ and

$$\nabla_{\bar{g}} \max(v,w) = \begin{cases} \nabla_{\bar{g}} v & : \text{if } v \geq w \\ \nabla_{\bar{g}} w & : \text{otherwise.} \end{cases}$$

Since $v = v^+ - v^-$, we have $\nabla_{\bar{g}} v = \nabla_{\bar{g}} v^+ - \nabla_{\bar{g}} v^-$, and therefore $\nabla v|_{\{v=0\}} = 0$ almost everywhere (this is essentially a result of Stampacchia [116], see also [69, Remark 2.4.26]). Also, if v_n, w_n are such that $v_n \to v$ and $w_n \to w$ in $H^1(\mathcal{C})$, then $\max(v_n, w_n) \to \max(v,w)$ in $H^1(\mathcal{C})$ [74, Lemma 1.22]. Therefore, $T_k(v) \in H^1(\mathcal{C})$ whenever $v \in H^1(\mathcal{C})$. Also $T_k(v) \to v$ in $H^1(\mathcal{C})$ as $k \to \infty$ and $T_k \colon H^1(\mathcal{C}) \to H^1(\mathcal{C})$ is continuous. If $v \in L^2_{H^1(\mathcal{C})}$, then $T_k(v) \in L^2_{H^1(\mathcal{C})}$ too, since $\phi_{\mathcal{C},-t} T_k(v(t)) = T_k(\phi_{\mathcal{C},-t} v(t))$ and $T_k(\tilde{v}) \in L^2(0,T;H^1(\mathcal{C}_0))$ whenever $\tilde{v} \in L^2(0,T;H^1(\mathcal{C}_0))$.

Clearly, all of this applies when we replace $\mathcal{C}$ with $\mathcal{C}_R$ but we can drop the requirement $f(0) = 0$.

4.4 The harmonic extension problems on evolving spaces

In this section, we shall consider the following two problems.

(1) Given $u \in L^2_{W^{1/2,2}}$, find $v \in L^2_{X(\mathcal{C})}$ such that

$$\begin{aligned} \Delta_{\bar{g}(t)} v(t) &= 0 \qquad \text{on } \mathcal{C}(t) \\ v(t,x,0) &= u(t,x) \end{aligned} \tag{4.21}$$

holds in the weak sense:

$$\begin{gathered} \int_0^T \int_{\mathcal{C}(t)} \overline{\nabla_{\bar{g}(t)}} v(t) \nabla_{\bar{g}(t)} \eta(t) = 0 \quad \text{for all } \eta \in L^2_{H^1}(\mathcal{C}) \text{ with } \mathbb{T}\eta = 0 \\ \overline{\mathbb{T}} v = u \quad \text{in } L^2_{W^{1/2,2}}. \end{gathered} \tag{4.22}$$

(2) Given $u \in L^2_{W^{1/2,2}}$, find $v \in L^2_{H^1(\mathcal{C}_R)}$ such that

$$\begin{aligned} \Delta_{\bar{g}(t)} v(t) &= 0 \qquad \text{on } \mathcal{C}_R(t) \\ v(t,x,0) &= u(t,x) \\ v(t,x,R) &= 0 \end{aligned} \tag{4.23}$$

holds in the weak sense:

$$\begin{aligned}\int_0^T \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)} v(t) \nabla_{\bar{g}(t)} \eta(t) = 0 \quad &\text{for all } \eta \in L^2_{H^1_0(\mathcal{C}_R)} \\ \mathbb{T}_{R,y=0} v = u \quad &\text{in } L^2_{W^{1/2,2}} \\ \mathbb{T}_{R,y=R} = 0.&\end{aligned} \tag{4.24}$$

As alluded to in the introduction, we study these problems in order to derive measurability in time of $\overline{\mathcal{E}}_t$ and $\overline{\mathcal{E}}_{R,t}$.

4.4.1 The harmonic extension of $u \in L^2_{W^{1/2,2}}$

Lemma 4.4.1. For every $u \in L^2_{W^{1/2,2}}$ such that $\int_{\Gamma_0} \phi_{-t} u(t) = 0$ for almost all t, there exists a $U \in L^2_{H^1(\mathcal{C})}$ such that $\mathbb{T}U = J_0^{(\cdot)} u(\cdot)$ and $\int_{\Gamma(t)} U(t,y) = 0$ for almost all t and all y.

Proof. Set $\tilde{u} = \phi_{-(\cdot)} u(\cdot) \in L^2(0,T;W^{1/2,2}(\Gamma_0))$ which has spatial mean value zero for a.a. t, and define

$$\tilde{U}(t) = \frac{1}{J_t^0} \mathcal{E}_0(\tilde{u}(t)).$$

We claim that $\tilde{U} \in L^2(0,T;H^1(\mathcal{C}_0))$. Observe that

$$\left\|\tilde{U}(t)\right\|^2_{L^2(\mathcal{C}_0)} = \int_0^\infty \int_{\Gamma_0} |(J_t^0)^{-1}|^2 |\mathcal{E}_0(\tilde{u}(t))|^2 \leq \frac{\left\|(J_t^0)^{-1}\right\|^2_{L^\infty(\Gamma_0)}}{2\lambda_1^{1/2}} \|\tilde{u}(t)\|^2_{L^2(\Gamma_0)},$$

and also

$$\begin{aligned}&\left\|\nabla_{\bar{g}} \tilde{U}(t)\right\|^2_{L^2(\mathcal{C}_0)} \\ &\quad\leq 2\left(\left\|(J_t^0)^{-1}\right\|^2_{L^\infty(\Gamma_0)} \|\nabla_{\bar{g}} \mathcal{E}_0(\tilde{u}(t))\|^2_{L^2(\mathcal{C}_0)} + \left\|\nabla_\Gamma (J_t^0)^{-1}\right\|^2_{L^\infty(\Gamma_0)} \|\mathcal{E}_0(\tilde{u}(t))\|^2_{L^2(\mathcal{C}_0)}\right) \\ &\quad\leq 2\left(\left\|(J_t^0)^{-1}\right\|^2_{L^\infty(\Gamma_0)} |\tilde{u}(t)|^2_{H(\Gamma_0)} + \frac{1}{2\lambda_1^{1/2}} \left\|\nabla_\Gamma (J_t^0)^{-1}\right\|^2_{L^\infty(\Gamma_0)} \|\tilde{u}(t)\|^2_{L^2(\Gamma_0)}\right),\end{aligned}$$

so certainly $\tilde{U} \in L^2(0,T;H^1(\mathcal{C}_0))$ (measurability of $t \mapsto \tilde{U}(t)$ can be inferred from consideration of Nemytskii maps). Set $U(t) := \phi_{\mathcal{C},t} \tilde{U}(t)$; this satisfies $(\mathbb{T}U)(t) = \mathcal{T}_t U(t) = \phi_{\Gamma,t} \mathcal{T}_0(\tilde{U}(t)) = \phi_{\Gamma,t}((J_t^0)^{-1}) u(t) = J_0^t u(t)$ since $\phi_{\Gamma,-t}(J_0^t) = 1/J_t^0$, and also, $\int_{\Gamma(t)} U(t) = \int_{\Gamma_0} \tilde{U}(t) J_t^0 = \int_{\Gamma_0} \mathcal{E}_0(\tilde{u}(t)) = 0$ as desired. □

Corollary 4.4.2. For every $u \in L^2_{W^{1/2,2}}$ with $\int_{\Gamma(t)} u(t) = 0$ for a.a. t, there exists a $U \in L^2_{H^1(\mathcal{C})}$ with $\mathbb{T}U = u$ and $\int_{\Gamma(t)} U(t,y) = 0$ a.e. t and for all y.

Proof. Let u be as stated and set $w(t) = u(t)/J_0^t$; note that $w \in L^2_{W^{1/2,2}}$ and $\fint_{\Gamma_0} \phi_{\Gamma,-t} w(t) = \fint_{\Gamma_0} \phi_{\Gamma,-t} u(t)/\phi_{\Gamma,-t} J_0^t = \fint_{\Gamma(t)} J_0^t u(t)/J_0^t = \fint_{\Gamma(t)} u(t) = 0$. So by the previous lemma, there exists a $U \in L^2_{H^1(\mathcal{C})}$ with $\mathbb{T}U = J_0^{(\cdot)} w(\cdot) = u$ and $\fint_{\Gamma(t)} U(t,y) = 0$. $\square$

Theorem 4.4.3 (Well-posedness of the harmonic extension problem in the space L^2_X)**.** There exists a map $\overline{\mathbb{E}}\colon L^2_{W^{1/2,2}} \to L^2_{X(\mathcal{C})}$ such that given $u \in L^2_{W^{1/2,2}}$, $v = \overline{\mathbb{E}}u$ is the unique weak solution of (4.21) satisfying (4.22) and for almost all t $\frac{1}{|\Gamma(t)|}\int_{\Gamma(t)} (\overline{\mathbb{E}}u)(t) = \bar{u}(t)$.

When $\overline{u(t)} = 0$ for a.a. t, we write the solution as $\mathbb{E}u$. The map $\overline{\mathbb{E}}$ satisfies $\overline{\mathbb{E}}u = \mathbb{E}(u - \bar{u}) + \bar{u}$.

Proof. First, suppose that $\overline{u(t)} = 0$ for a.e. t. Let us transform the equation to one with zero initial trace. By the previous corollary, there exists a $U \in L^2_{H^1(\mathcal{C})}$ with $\mathbb{T}U = u$ and crucially $\overline{U(t,y)} = 0$ for a.a. t and all y. Set $d := v - U \in L^2_{H^1(\mathcal{C})}$ which satisfies

$$\begin{aligned} \Delta_{\bar{g}} d &= -\Delta_{\bar{g}} U \\ \mathbb{T}d &= 0. \end{aligned}$$

Define $\hat{X} = \{d \in L^2_{H^1(\mathcal{C})} \mid \mathbb{T}d = 0 \text{ and } \overline{d(t,y)} = 0 \text{ for all } y \text{ and a.a. } t\}$; being a closed linear subspace of $L^2_{H^1(\mathcal{C})}$ (thanks to the continuity of $\mathbb{T}$ and $y \mapsto \overline{d(t,y)}$) this is a reflexive separable Banach space. Define the functional $J\colon \hat{X} \to \mathbb{R}$ by

$$J(d) = \frac{1}{2}\int_0^T \int_{\mathcal{C}(t)} |\nabla_{\bar{g}(t)} d(t)|^2 + \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} U(t) \nabla_{\bar{g}(t)} d(t).$$

We have that J is coercive because, using Poincaré's inequality,

$$\begin{aligned} J(d) &\geq C_1 \int_0^T \int_{\mathcal{C}(t)} d(t)^2 + |\nabla_\Gamma d(t)|^2 + d_y(t)^2 - C_\epsilon |\nabla_{\bar{g}(t)} U(t)|^2 - \frac{\epsilon}{2} |\nabla_{\bar{g}(t)} d(t)|^2 \\ &\geq C_2 \int_0^T \int_{\mathcal{C}(t)} d(t)^2 + |\nabla_{\bar{g}(t)} d(t)|^2 - C_3 \int_0^T \int_{\mathcal{C}(t)} |\nabla_{\bar{g}(t)} U(t)|^2 \\ &= C_2 \|d\|^2_{L^2_{H^1(\mathcal{C})}} - C_3 \left\|\nabla_{\bar{g}(t)} U\right\|^2_{L^2_{L^2(\mathcal{C})}} \end{aligned}$$

which clearly implies $J(d_n) \to \infty$ whenever $\|d_n\|_{L^2_{H^1(\mathcal{C})}} \to \infty$. Since $d \mapsto J(d)$ is continuous, it is lower semi-continuous. For the convexity, we have

$$
\begin{aligned}
&J(\lambda d_1 + (1-\lambda)d_2) \\
&= \frac{1}{2}\|\lambda d_1 + (1-\lambda)d_2\|^2_{L^2_{X(\mathcal{C})}} + \int_0^T \int_{\mathcal{C}(t)} \lambda \nabla_{\bar{g}(t)} U(t) \nabla_{\bar{g}(t)} d_1(t) \\
&\quad + \int_{\mathcal{C}(t)} (1-\lambda)\nabla_{\bar{g}(t)} U(t) \nabla_{\bar{g}(t)} d_2(t) \\
&\leq \frac{1}{2}(\lambda \|d_1\|_{L^2_{X(\mathcal{C})}} + (1-\lambda)\|d_2\|_{L^2_{X(\mathcal{C})}})^2 + \lambda \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} U(t) \nabla_{\bar{g}(t)} d_1(t) \\
&\quad + (1-\lambda)\int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} U(t) \nabla_{\bar{g}(t)} d_2(t) \\
&< \frac{1}{2}(\lambda \|d_1\|^2_{L^2_{X(\mathcal{C})}} + (1-\lambda)\|d_2\|^2_{L^2_{X(\mathcal{C})}}) + \lambda \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} U(t) \nabla_{\bar{g}(t)} d_1(t) \\
&\quad + (1-\lambda)\int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} U(t) \nabla_{\bar{g}(t)} d_2(t) \qquad \text{(since } x \mapsto x^2 \text{ is strictly convex)} \\
&= \lambda J(d_1) + (1-\lambda)J(d_2).
\end{aligned}
$$

By Theorem 5.25 of [43], this problem has a unique minimiser d and it satisfies $J'(d,w) = 0$ for all $w \in \hat{X}$. Since

$$
\begin{aligned}
J'(d,w) &:= \lim_{\lambda \to 0} \frac{J(d+\lambda w) - J(d)}{\lambda} \\
&= \lim_{\lambda \to 0} \frac{1}{2\lambda} \int_0^T \int_{\mathcal{C}(t)} 2\lambda \nabla_{\bar{g}(t)} d(t) \nabla_{\bar{g}(t)} w(t) + \lambda^2 |\nabla_{\bar{g}(t)} w(t)|^2 \\
&\quad + \frac{1}{\lambda}\int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} U(t) \lambda \nabla_{\bar{g}(t)} w(t) \\
&= \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} d(t) \nabla_{\bar{g}(t)} w(t) + \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} U(t) \nabla_{\bar{g}(t)} w(t),
\end{aligned}
$$

and recalling that $v = d + U$, we find that $v \in L^2_{H^1(\mathcal{C})}$ with $\mathbb{T}v = u$ and $\bar{v} = \bar{d} + \bar{U} = 0$ satisfies

$$\int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} v(t) \nabla_{\bar{g}(t)} w(t) = 0$$

for all $w \in \hat{X}$. We want to remove the mean value zero condition on the test functions. To that end, let $\eta \in L^2_{H^1(\mathcal{C})}$ with $\mathbb{T}\eta = 0$ and set $w(t) = \eta(t) - \bar{\eta}(t)$ (this satisfies $\mathbb{T}w = 0$ and $\bar{w}(t) = \bar{\eta}(t) - \bar{\eta}(t) = 0$ so is admissible):

$$
\begin{aligned}
0 &= \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} v(t) \nabla_{\bar{g}(t)} \eta(t) - \int_0^T \int_{\mathcal{C}(t)} v_y(t) \bar{\eta}_y(t) \\
&= \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} v(t) \nabla_{\bar{g}(t)} \eta(t) - \int_0^T \int_0^\infty \int_{\Gamma(t)} v_y(t) \partial_y \left(\frac{1}{|\Gamma(t)|} \int_{\Gamma(t)} \eta(t) \right) \\
&= \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} v(t) \nabla_{\bar{g}(t)} \eta(t) - \int_0^T \int_0^\infty \partial_y \left(\frac{1}{|\Gamma(t)|} \int_{\Gamma(t)} v(t) \right) \partial_y \left(\int_{\Gamma(t)} \eta(t) \right) \\
&= \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} v(t) \nabla_{\bar{g}(t)} \eta(t)
\end{aligned}
$$

since $\bar{v}(t) = 0$ for a.a. t and all y. This settles the problem for the case $\overline{u(t,y)} = 0$. For general $u \in L^2_{W^{1/2,2}}$, define $\overline{\mathbb{E}}u := \mathbb{E}(u - \bar{u}) + \bar{u}$, which is such that $\overline{\mathbb{E}}u \in L^2_{X(\mathcal{C})}$ and $(\overline{\mathbb{T}\mathbb{E}}u)(t) = u(t)$. Finally, it satisfies $\frac{1}{|\Gamma(t)|} \int_{\Gamma(t)} \overline{\mathbb{E}}u(t) = \bar{u}(t)$ and

$$\int_0^T \int_{\mathcal{C}(t)} \overline{\nabla_{\bar{g}(t)}} (\overline{\mathbb{E}}u)(t) \nabla_{\bar{g}(t)} \eta(t) = \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} (\mathbb{E}(u - \bar{u}))(t) \nabla_{\bar{g}(t)} \eta = 0$$

for all $\eta \in L^2_{H^1(\mathcal{C})}$ with $\mathbb{T}\eta = 0$. $\square$

We need to elucidate the link between $\mathbb{E}$ and the family of maps $\{\mathcal{E}_t\}_{t \in [0,T]}$.

Lemma 4.4.4. Let $u \in L^2_{W^{1/2,2}}$. For almost all t, $(\overline{\mathbb{E}}u)(t) = \overline{\mathcal{E}}_t u(t)$.

Proof. We begin with

$$\int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} (\overline{\mathbb{E}}u)(t) \nabla_{\bar{g}(t)} \eta(t) = 0 \quad \text{for all } \eta \in L^2_{H^1(\mathcal{C})} \text{ with } \mathbb{T}\eta = 0.$$

Pick $\psi \in C_c^\infty(0,T)$ and $v_0 \in H^1(\mathcal{C}_0)$ with $\mathcal{T}_0 v_0 = 0$, then $\psi \phi_{\mathcal{C},t} v_0 \in L^2_{H^1(\mathcal{C})}$ with $\mathbb{T}(\psi \phi_{\mathcal{C},t} v_0) = 0$, so it is an admissible test function in (4.22) and testing with it gives

$$\int_0^T \psi(t) \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} (\overline{\mathbb{E}}u)(t) \nabla_{\bar{g}(t)} \phi_{\mathcal{C},t} v_0 = 0 \quad \text{for all } v_0 \in H^1(\mathcal{C}_0) \text{ with } \mathcal{T}_0 v_0 = 0$$

which implies

$$\int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} (\overline{\mathbb{E}}u)(t) \nabla_{\bar{g}(t)} \phi_{\mathcal{C},t} v_0 = 0 \quad \text{for all } v_0 \in H^1(\mathcal{C}_0) \text{ with } \mathcal{T}_0 v_0 = 0$$

for almost all t. By the homeomorphism properties of $\phi_{\mathcal{C},t}$, this is same as

$$\int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)}(\overline{\mathbb{E}}u)(t)\nabla_{\bar{g}(t)}v_t = 0 \quad \text{for all } v_t \in H^1(\mathcal{C}(t)) \text{ with } \mathcal{T}_t v_t = 0, \text{ for almost all } t,$$

and since also $\overline{\mathcal{T}}_t(\overline{\mathbb{E}}u(t)) = u(t)$, we have $(\overline{\mathbb{E}}u)(t) = \overline{\mathcal{E}}_t u(t)$ by uniqueness of solutions to the harmonic extension problem (Theorem 4.2.4). $\square$

Lemma 4.4.5. For all $u \in L^2_{W^{1/2,2}}$ with $\bar{u} = 0$,

$$\|\mathbb{E}u\|_{L^2_{L^2(\mathcal{C})}} \leq C \|u\|_{L^2_{L^2}}$$

$$\|\nabla_{\bar{g}}\mathbb{E}u\|_{L^2_{L^2(\mathcal{C})}} \leq C \|u\|_{L^2_{W^{1/2,2}}}.$$

Proof. Thanks to the the previous lemma, we can simply use the bound (4.11) and Lemma 4.3.6:

$$\begin{aligned}
\|\mathbb{E}u\|^2_{L^2_{L^2(\mathcal{C})}} &= \int_0^T \|\mathcal{E}_t u(t)\|^2_{L^2(\mathcal{C}(t)} \leq C_1 \int_0^T \|u(t)\|^2_{L^2(\Gamma(t))} \\
\|\nabla_{\bar{g}}\mathbb{E}u\|^2_{L^2_{L^2(\mathcal{C})}} &= \int_0^T \left\|\nabla_{\bar{g}(t)}\mathcal{E}_t u(t)\right\|^2_{L^2(\mathcal{C}(t)} \leq C_2 \int_0^T \|u(t) - \bar{u}(t)\|^2_{W^{1/2,2}(\Gamma(t))},
\end{aligned}$$

where we also used the eigenvalue estimate (A_λ) in the first estimate. $\square$

4.4.2 The truncated harmonic extension of $u \in L^2_{W^{1/2,2}}$

Theorem 4.4.6 (Well-posedness of the truncated harmonic extension problem in the space $L^2_{H^1(\mathcal{C}_R)}$)**.** There exists a map $\overline{\mathbb{E}}_R \colon L^2_{W^{1/2,2}} \to L^2_{H^1(\mathcal{C}_R)}$ such that given $u \in L^2_{W^{1/2,2}}$, $\overline{\mathbb{E}}_R u$ is the unique weak solution of (4.23) satisfying (4.24) and for almost all t $\frac{1}{|\Gamma(t)|}\int_{\Gamma(t)}(\overline{\mathbb{E}}u)(t) = \bar{u}(t)$.

When $\overline{u(t)} = 0$ for a.a. t, we write the solution as $\mathbb{E}_R u$.

Proof. We transform (4.23) into a problem with zero boundary conditions by setting $w = v - \frac{R-y}{R}\overline{\mathbb{E}}u \in L^2_{H^1(\mathcal{C}_R)}$ where $v = \overline{\mathbb{E}}_R u$; then w satisfies

$$\begin{aligned}
\Delta_{\bar{g}(t)}w(t) &= -\Delta_{\bar{g}(t)}\left(\frac{R-y}{R}\overline{\mathbb{E}}u(t)\right) \quad \text{on } \mathcal{C}_R(t) \\
w(t,x,0) &= 0 \\
w(t,x,R) &= 0,
\end{aligned}$$

which has a unique solution $w \in L^2_{H^1_0(\mathcal{C}_R)}$ satisfying

$$\int_0^T \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)} w(t) \nabla_{\bar{g}(t)} \eta(t) = -\int_0^T \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)} \left(\frac{R-y}{R} \overline{\mathcal{E}}_t(u(t)) \right) \nabla_{\bar{g}(t)} \eta(t) \tag{4.25}$$

for all $\eta \in L^2_{H^1_0(\mathcal{C}_R)}$. Indeed, define a bilinear form $a \colon L^2_{H^1_0(\mathcal{C}_R)} \times L^2_{H^1_0(\mathcal{C}_R)} \to \mathbb{R}$ by the left hand side of the above equality. It is clearly bounded, and coercivity follows from Poincaré's inequality which holds for the following reason. Since[2]

$$\|\eta\|^2_{L^2_{H^1_0(\mathcal{C}_R)}} = \int_0^T \int_{\Gamma(t)} \int_0^R |\eta(t)|^2 + |\nabla_\Gamma \eta(t)|^2 + |\eta_y(t)|^2 < \infty$$

it follows that for a.a. t, a.a. x, $\eta(t, x, \cdot) \in H^1_0(0, R)$ (recall that η vanishes at $y = R$ and $y = 0$). Thus the Poincaré inequality implies

$$\int_0^R |\eta(t, x)|^2 \leq C_P \int_0^R |\eta_y(t, x)|^2$$

for a.a. t and a.a. x. Using this fact in the definition of the norm of $L^2_{H^1_0(\mathcal{C}_R)}$ gives $\|\eta\|_{L^2_{H^1_0(\mathcal{C}_R)}} \leq C \|\nabla_{\bar{g}} \eta\|_{L^2_{L^2(\mathcal{C}_R)}}$ so that $\|\nabla_{\bar{g}} \cdot\|_{L^2_{L^2(\mathcal{C}_R)}}$ is an equivalent norm on $L^2_{H^1_0(\mathcal{C}_R)}$. Defining $l \colon L^2_{H^1_0(\mathcal{C}_R)} \to \mathbb{R}$ by the right hand side of (4.25), we can also see that l is in the dual space of $L^2_{H^1_0(\mathcal{C}_R)}$:

$$\begin{aligned}
l(\eta) &= \int_0^T \int_0^R \int_{\Gamma(t)} \frac{R-y}{R} \nabla_{\bar{g}(t)} \overline{\mathcal{E}}_t(u(t)) \nabla_{\bar{g}(t)} \eta(t) + \overline{\mathcal{E}}_t(u(t)) \nabla_{\bar{g}(t)} \left(\frac{R-y}{R} \right) \nabla_{\bar{g}(t)} \eta(t) \\
&\leq \int_0^T \int_0^R \int_{\Gamma(t)} 2 |\nabla_{\bar{g}(t)} \overline{\mathcal{E}}_t(u(t))| |\nabla_{\bar{g}(t)} \eta(t)| + \frac{1}{R} |\overline{\mathcal{E}}_t(u(t))| |\partial_y \eta(t)| \\
&\leq C(R) \left(\left\|\nabla_{\bar{g}} \overline{\mathbb{E}} u\right\|_{L^2_{L^2(\mathcal{C}_R)}} + \left\|\overline{\mathbb{E}} u\right\|_{L^2_{L^2(\mathcal{C}_R)}} \right) \|\nabla_{\bar{g}} \eta\|_{L^2_{L^2(\mathcal{C}_R)}} \\
&\leq C(R) \left\|\overline{\mathbb{E}} u\right\|_{L^2_{H^1(\mathcal{C}_R)}} \|\nabla_{\bar{g}} \eta\|_{L^2_{L^2(\mathcal{C}_R)}}.
\end{aligned}$$

Therefore, Lax–Milgram gives the result. It then follows that $\overline{\mathbb{E}}_R u := w + \frac{R-y}{R} \overline{\mathbb{E}}(u) \in L^2_{H^1(\mathcal{C}_R)}$ satisfies $\mathbb{T}_{R,y=0} v = u$, $\mathbb{T}_{R,y=R} = 0$ and the weak formulation in (4.24). $\square$

Lemma 4.4.7. Let $u \in L^2_{W^{1/2,2}}$. For almost all t, $(\overline{\mathbb{E}}_R u)(t) = \overline{\mathcal{E}_{R,t}} u(t)$.

[2] The interchange of integrals over $[0, R]$ and $\Gamma(t)$ is justified for the following reason. Suppose $w \in H^1(\mathcal{C}_R(t))$. This means that w, $\nabla_\Gamma w$ and w_y belong to $L^2((0, R) \times \Gamma(t))$, so are measurable in the product space, and Fubini–Tonelli allows us to integrate in any order.

Proof. We begin with

$$\int_0^T \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)}(\overline{\mathbb{E}}_R u)(t) \nabla_{\bar{g}(t)} \eta(t) = 0 \quad \text{for all } \eta \in L^2_{H^1_0(\mathcal{C}_R)}.$$

Pick $\psi \in C_c^\infty(0,T)$ and $v_0 \in H^1_0(\mathcal{C}_R(0))$ with $\mathcal{T}_0 v_0 = 0$, then $\psi\phi_{\mathcal{C},t}v_0 \in L^2_{H^1_0(\mathcal{C}_R)}$ with $\mathbb{T}(\psi\phi_{\mathcal{C},t}v_0) = 0$, so it is an admissible test function in (4.24) and testing with it gives

$$\int_0^T \psi(t) \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)}(\overline{\mathbb{E}}_R u)(t) \nabla_{\bar{g}(t)} \phi_{\mathcal{C},t} v_0 = 0 \quad \text{for all } v_0 \in H^1_0(\mathcal{C}_R(0))$$

which implies

$$\int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)}(\overline{\mathbb{E}}_R u)(t) \nabla_{\bar{g}(t)} \phi_{\mathcal{C},t} v_0 = 0 \quad \text{for all } v_0 \in H^1_0(\mathcal{C}_R(0)) \text{ for almost all } t.$$

By the homeomorphism properties of $\phi_{\mathcal{C},t}$, this is same as

$$\int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)}(\overline{\mathbb{E}}_R u)(t) \nabla_{\bar{g}(t)} v_t = 0 \quad \text{for all } v_t \in H^1_0(\mathcal{C}_R(t)) \text{ for almost all } t,$$

and since also $\mathcal{T}_{R,t,y=0}((\overline{\mathbb{E}}_R u)(t)) = u(t)$ and $\mathcal{T}_{R,t,y=R}((\overline{\mathbb{E}}_R u)(t)) = 0$, we have $(\overline{\mathbb{E}}_R u)(t) = \overline{\mathcal{E}_{R,t}} u(t)$ by uniqueness of solutions to the truncated harmonic extension problem. $\square$

Lemma 4.4.8. When $R \geq 1$, we have

$$\left\|\nabla_{\bar{g}} \overline{\mathbb{E}}_R u\right\|_{L^2_{L^2(\mathcal{C}_R)}} \leq C \left\|u\right\|_{L^2_{H^{1/2}}}.$$

where C is independent of R.

Proof. This easily follows from the previous lemma and Lemma 4.3.7 (since $R > 1$, the dependence on R given in the latter lemma disappears). $\square$

A third way to interpret the map $\mathscr{Z}_R$ from Definition 4.2.19 is as a map $\mathscr{Z}_R \colon \{\eta \in L^2_{H^1(\mathcal{C}_R)} \mid \mathbb{T}_{R,y=R}\eta = 0\} \to L^2_{H^1(\mathcal{C})}$, and again this map preserves norms.

Lemma 4.4.9. We have

$$\mathscr{Z}_R \overline{\mathbb{E}}_R u \to \overline{\mathbb{E}} u \quad \text{in } L^2_{X(\mathcal{C})}.$$

Proof. Lemma 4.2.20 gives for almost all t

$$\int_0^\infty \int_{\Gamma(t)} |\nabla_{\bar{g}(t)} \mathscr{L}_R \overline{\mathcal{E}}_{R,t} u(t) - \nabla_{\bar{g}(t)} \overline{\mathcal{E}}_t u(t)|^2 \to 0$$

and so it suffices to find an integrable in time uniform in R bound on the above expression. We have, using Lemma 4.3.7,

$$\begin{aligned}\int_0^\infty \int_{\Gamma(t)} |\nabla_{\bar{g}(t)} \mathscr{L}_R \overline{\mathcal{E}}_{R,t} u(t)|^2 &= \int_0^R \int_{\Gamma(t)} |\nabla_{\bar{g}(t)} \overline{\mathcal{E}}_{R,t} u(t)|^2 \\ &\leq C_1 \|u - \bar{u}\|^2_{W^{1/2}(\Gamma(t))} + \frac{C_2}{R^2} \|u - \bar{u}\|^2_{L^2(\Gamma(t))} + \frac{2|\bar{u}|^2}{R} |\Gamma|\end{aligned}$$

and the RHS is integrable over time and if $R \geq 1$ it is uniform. Then the DCT gives the result. $\square$

4.5 The non-degenerate problem: proof of Theorem 4.1.6

Let $\beta\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying (A_β) on p. 113. We will prove Theorem 4.1.6 in this section, that of the well-posedness of problem $(\mathbf{P}_\beta)$.

4.5.1 Existence of solutions to the truncated problem

In this subsection, we will prove the following theorem. For easier reading, we will shorten the duality products $\langle \cdot, \cdot \rangle_{W^{-1/2,2}(\Gamma(t)), W^{1/2,2}(\Gamma(t))}$ to $\langle \cdot, \cdot \rangle$ (an abuse of notation) and $\langle \cdot, \cdot \rangle_{W^{-1/2,2}(\Gamma_0), W^{1/2,2}(\Gamma_0)}$ to $\langle \cdot, \cdot \rangle_0$.

Theorem 4.5.1. For each R sufficiently large, there exists a unique weak solution $u_R \in \mathbb{W}(W^{1/2,2}, W^{-1/2,2})$ to $(\mathbf{P}_{\beta R})$ with $\nabla_{\bar{g}} \overline{\mathbb{E}}_R(\beta(u_R)) \in L^2_{L^2(\mathcal{C}_R)}$ and $u_R(0) = u_0$ satisfying

$$\begin{aligned}\int_0^T \langle \dot{u}_R(t), \eta(t) \rangle + \int_0^T \int_{\Gamma(t)} u_R(t) \eta(t) \nabla_\Gamma \cdot \mathbf{w} \\ + \int_0^T \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)} \overline{\mathcal{E}}_{R,t}(\beta(u_R(t))) \nabla_{\bar{g}(t)} (E_R \eta)(t) = 0\end{aligned}$$

for all $\eta \in L^2_{W^{1/2,2}}$, where $E_R \eta \in L^2_{H^1(\mathcal{C}_R)}$ satisfies $\mathbb{T}_{R,y=0} E_R \eta = \eta$ and $\mathbb{T}_{R,y=R} E_R \eta = 0$.

Define $a_R(t;\cdot,\cdot)\colon W^{1/2,2}(\Gamma(t)) \times W^{1/2,2}(\Gamma(t)) \to \mathbb{R}$ by

$$a_R(t;u,\eta) = \int_0^R \int_{\Gamma(t)} \nabla_{\bar{g}(t)} \overline{\mathcal{E}}_{R,t}(\beta(u)) \nabla_{\bar{g}(t)} E_R(t)\eta$$

where $E_R(t)\colon W^{1/2,2}(\Gamma(t)) \to H^1(\mathcal{C}_R(t))$ is an (arbitrary) extension that satisfies $\mathcal{T}_{R,t,y=0}(E_R(t)\eta) = \eta$ and $\mathcal{T}_{R,t,y=R}(E_R(t)\eta) = 0$; the choice of E_R does not matter (see Remark 4.2.18). We hide the subscript R in u_R and write simply u for simpler notation. We define $\tilde{u} = \phi_{\Gamma,-(\cdot)}u$ and $\tilde{\eta} = \phi_{\Gamma,-(\cdot)}\eta$ and rewrite the equation

$$\int_{\Gamma(t)} \dot{u}(t)\eta(t) + \int_{\Gamma(t)} u(t)\eta(t)\nabla_\Gamma \cdot \mathbf{w} + a_R(t;u(t),\eta(t)) = 0$$

in terms of $\tilde{u}$ and $\tilde{\eta}$:

$$\int_{\Gamma_0} \tilde{u}'(t)\tilde{\eta}(t)J_t^0 + \int_{\Gamma_0} \tilde{u}(t)\tilde{\eta}(t)J_t^0\phi_{-t}(\nabla_\Gamma \cdot \mathbf{w}) + a_R(t;\phi_t\tilde{u}(t),\phi_t\tilde{\eta}(t)) = 0$$

(for ease of reading we wrote ϕ_t instead of $\phi_{\Gamma,t}$). Now substitute $\tilde{\psi} = \tilde{\eta}J_t^0$ and use $1/\phi_t J_t^0 = J_0^t$:

$$\int_{\Gamma_0} \tilde{u}'(t)\tilde{\psi}(t) + \int_{\Gamma_0} \tilde{u}(t)\tilde{\psi}(t)\phi_{-t}(\nabla_\Gamma \cdot \mathbf{w}) + a_R(t;\phi_t\tilde{u}(t), J_0^t\phi_t\tilde{\psi}(t)) = 0.$$

We seek a Galerkin approximation of this equation. Let $\{b_j\}$ be an orthonormal basis of $L^2(\Gamma_0)$ that is orthogonal in $W^{1/2,2}(\Gamma_0)$ and consider the system

$$\begin{aligned}\int_{\Gamma_0} \tilde{u}_n'(t)b_j + \int_{\Gamma_0} \tilde{u}_n(t)b_j\phi_{-t}(\nabla_\Gamma \cdot \mathbf{w}) + a_R(t;\phi_t\tilde{u}_n(t), J_0^t\phi_t b_j) = 0 \quad \forall j = 1,...,n\\ \tilde{u}_n(0) = \tilde{u}_{0n}\end{aligned} \tag{4.26}$$

for an ansatz $\tilde{u}_n(t) = \sum_{i=1}^n \alpha_i(t)b_i$ with unknown coefficients $\alpha_i = \alpha_i^n$ and $\tilde{u}_{0n} \in V_n(0) := \operatorname{span}\{b_1,...,b_n\}$ such that $\tilde{u}_{0n} \to u_0$ in $W^{1/2,2}(\Gamma_0)$ and $\|\tilde{u}_{0n}\|_{W^{1/2,2}(\Gamma_0)} \leq C\|u_0\|_{W^{1/2,2}(\Gamma_0)}$.

Remark 4.5.2. We have pulled back the equation onto a reference domain in order to facilitate the procurement of a bound on $\tilde{u}_n'$, which is needed for a strong convergence result to pass to the limit in the nonlinear term. This transformation to the reference domain Γ_0 could have been avoided if we knew that the orthogonal projection operator $P_n^t\colon L^2(\Gamma(t)) \to V_n(t) := \phi_t(V_n(0))$ defined by

$$(P_n^t u - u, v_n)_{L^2(\Gamma(t))} = 0 \quad \text{for all } v_n \in V_n(t)$$

was bounded as a map $P_n^t\colon V(t) \to V(t)$. Such a bound is true when $t = 0$ because of the special choice of basis functions, but for arbitrary t the desired bound appears elusive. Of course, such a result would be of fundamental use generally in parabolic equations on evolving domains.

The following (surprisingly non-trivial) lemma is useful below. The proof of the continuity is the same as in Lemma 2.5 of [22], adapted to our setting; we give the proof in the appendix for convenience.

Lemma 4.5.3. For a sufficiently smooth hypersurface Γ, if $\beta\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz with $\beta(0) = 0$, then the superposition map $\beta\colon W^{1/2,2}(\Gamma) \to W^{1/2,2}(\Gamma)$ is sequentially continuous and satisfies

$$\|\beta(u)\|_{W^{1/2,2}(\Gamma)} \leq \mathrm{Lip}(\beta)\, \|u\|_{W^{1/2,2}(\Gamma)} \quad \text{for all } u \in W^{1/2,2}(\Gamma).$$

Lemma 4.5.4. The Galerkin equation (4.26) has a solution $\tilde{u}_n \in H^1(0,T;V_n(0))$.

Proof. The equation (4.26) leads to

$$\alpha_j'(t) + \sum_{i=1}^n \alpha_i(t) \int_{\Gamma_0} b_i b_j \phi_{-t}(\nabla_\Gamma \cdot \mathbf{w}) + a_R(t; \sum_{i=1}^n \alpha_i(t)\phi_t b_i, J_0^t \phi_t b_j) = 0 \quad \forall j = 1, ..., n. \tag{4.27}$$

Let $\boldsymbol{\alpha}(t) = (\alpha_1(t), ..., \alpha_n(t))^\intercal$, $\mathbf{b}(t) = (\phi_t b_1, ..., \phi_t b_n)^\intercal$, define the matrix $(\mathbf{W}(t))_{ij} = \int_{\Gamma_0} b_j b_i \phi_{-t}(\nabla_\Gamma \cdot \mathbf{w})$ and vector

$$\mathbf{a}(t,\boldsymbol{\alpha}) = \begin{pmatrix} a_R(t; \sum_{i=1}^n \alpha_i \phi_t b_i, J_0^t \phi_t b_1) \\ \vdots \\ a_R(t; \sum_{i=1}^n \alpha_i \phi_t b_i, J_0^t \phi_t b_n) \end{pmatrix} = \begin{pmatrix} a_R(t; \boldsymbol{\alpha} \cdot \boldsymbol{b}(t), J_0^t \phi_t b_1) \\ \vdots \\ a_R(t; \boldsymbol{\alpha} \cdot \boldsymbol{b}(t), J_0^t \phi_t b_n) \end{pmatrix}.$$

The system of equations (4.27) is then written as

$$\boldsymbol{\alpha}'(t) = F(t, \boldsymbol{\alpha}(t)) := -\mathbf{W}(t)\boldsymbol{\alpha}(t) - \mathbf{a}(t, \boldsymbol{\alpha}(t))$$

for $F\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ defined by the right hand side.

Checking the Carathéodory condition: measurability We need to show that $t \mapsto F(t, \boldsymbol{\alpha})$ is measurable for fixed $\boldsymbol{\alpha} \in \mathbb{R}^n$. The term with the matrix is clear.

For the other term, consider

$$
\begin{aligned}
&a_R(t;\boldsymbol{\alpha}\cdot\boldsymbol{b}(t), J_0^t\phi_t b_j)\\
&\quad= \int_0^R\int_{\Gamma(t)} \nabla_{\Gamma(t)}\overline{\mathcal{E}}_{R,t}(\beta(\boldsymbol{\alpha}\cdot\boldsymbol{b}(t)))\nabla_{\Gamma(t)}E_R(t)(J_0^t\phi_t b_j)\\
&\qquad+\partial_y\overline{\mathcal{E}}_{R,t}(\beta(\boldsymbol{\alpha}\cdot\boldsymbol{b}(t)))\partial_y E_R(t)(J_0^t\phi_t b_j)\\
&\quad= \int_{\mathcal{C}_R(0)} J_t^0\nabla_{\Gamma(s)}\phi_{-t}[\overline{\mathcal{E}}_{R,t}(\beta(\boldsymbol{\alpha}\cdot\boldsymbol{b}(t)))](\mathbf{D}\Phi_t^0)^{-1}(\mathbf{D}\Phi_t^0)^{-\intercal}\nabla_{\Gamma(s)}\phi_{-t}[E_R(t)(J_0^t\phi_t b_j)]\\
&\qquad+\int_{\mathcal{C}_R(0)} J_t^0\partial_y\phi_{-t}\overline{\mathcal{E}}_{R,t}(\beta(\boldsymbol{\alpha}\cdot\boldsymbol{b}(t)))\partial_y\phi_{-t}E_R(t)(J_0^t\phi_t b_j),
\end{aligned}
$$

and we know that $\overline{\mathcal{E}}_{R,t}(\beta(\boldsymbol{\alpha}\cdot\boldsymbol{b}(t))) = \overline{\mathbb{E}}_R(\beta(\boldsymbol{\alpha}\cdot\boldsymbol{b}))(t)$ for a.a. t (Lemma 4.4.7), and the pullback of the latter is measurable as a function of t since $\overline{\mathbb{E}}_R(\beta(\boldsymbol{\alpha}\cdot\boldsymbol{b}))\in L^2_{H^1(\mathcal{C}_R)}$ as $\beta(\boldsymbol{\alpha}\cdot\boldsymbol{b})\in L^2_{W^{1/2,2}}$. The same argument can be used to deal with the $E_R(t)$ term.

Checking the Carathéodory condition: continuity Now suppose that $\boldsymbol{\alpha}^j\to\boldsymbol{\alpha}$ in $\mathbb{R}^n$. We see that

$$
\begin{aligned}
&\left\|\mathbf{a}(t,\boldsymbol{\alpha}^{\boldsymbol{j}})-\mathbf{a}(t,\boldsymbol{\alpha})\right\|^2_{\mathbb{R}^n}\\
&\quad=\sum_i\left|\int_0^R\int_{\Gamma(t)}\nabla_{\bar{g}(t)}[\overline{\mathcal{E}}_{R,t}(\beta(\boldsymbol{\alpha}^j\cdot\boldsymbol{b}(t)))-\overline{\mathcal{E}}_{R,t}(\beta(\boldsymbol{\alpha}\cdot\boldsymbol{b}(t)))]\nabla_{\bar{g}(t)}E_R(t)(J_0^t\phi_t b_i)\right|^2\\
&\quad\leq\sum_i\left(\int_0^R\int_{\Gamma(t)}|\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}[\beta(\boldsymbol{\alpha}^j\cdot\boldsymbol{b}(t))-\beta(\boldsymbol{\alpha}\cdot\boldsymbol{b}(t))]\nabla_{\bar{g}(t)}E_R(t)(J_0^t\phi_t b_i)|\right)^2\\
&\quad\leq\sum_i\left(C(R)\left\|\beta(\boldsymbol{\alpha}^j\cdot\boldsymbol{b}(t))-\beta(\boldsymbol{\alpha}\cdot\boldsymbol{b}(t))\right\|_{W^{1/2,2}(\Gamma(t))}\left\|\nabla_{\bar{g}(t)}E_R(t)(J_0^t\phi_t b_i)\right\|_{L^2(\mathcal{C}_R)}\right)^2
\end{aligned}
$$

by Lemma 4.3.7 and this tends to zero by Lemma 4.5.3 since $\boldsymbol{\alpha}^j\cdot\boldsymbol{b}(t)\to\boldsymbol{\alpha}\cdot\boldsymbol{b}(t)$ in $W^{1/2,2}(\Gamma(t))$:

$$\left\|\boldsymbol{\alpha}^j\cdot\boldsymbol{b}(t)-\boldsymbol{\alpha}\cdot\boldsymbol{b}(t)\right\|_{W^{1/2,2}(\Gamma(t))}\leq\sum_{i=1}^n|\boldsymbol{\alpha}_i^j-\boldsymbol{\alpha}_i|\left\|\boldsymbol{b}_i(t)\right\|_{W^{1/2,2}(\Gamma(t))}\to 0.$$

This implies that $\boldsymbol{\alpha}\mapsto\mathbf{a}(t,\boldsymbol{\alpha})$ is continuous and so $t\mapsto F(t,\boldsymbol{\alpha})$ is a Carathéodory function.

The uniform bound that we shall derive in the next subsection shows that $\|\boldsymbol{\alpha}(t)\|_{\mathbb{R}^n}\leq c$ for all t if $\boldsymbol{\alpha}$ satisfies the ODE (4.27). We also need to show that there exists $f\in L^1(0,T)$ with $\|F(t,\boldsymbol{\alpha})\|_{\mathbb{R}^n}\leq f(t)$ for every $\boldsymbol{\alpha}\in\{\boldsymbol{\alpha}\in\mathbb{R}^n\mid\|\boldsymbol{\alpha}\|_{\mathbb{R}^n}\leq 2c\}$. Let $\|\cdot\|_F$ denote the Frobenius matrix norm (i.e. the Euclidean norm) on $\mathbb{R}^n\times\mathbb{R}^n$.

We have

$$\frac{1}{2}\|F(t,\boldsymbol{\alpha})\|_{\mathbb{R}^n}^2 \leq \|\boldsymbol{W}(t)\|_F^2\,\|\boldsymbol{\alpha}\|_{\mathbb{R}^n}^2 + \|\mathbf{a}(t,\boldsymbol{\alpha})\|_{\mathbb{R}^n}^2$$

(because the Frobenius norm is compatible with the Euclidean vector norm) and note that

$$\begin{aligned}
\|\mathbf{a}(t,\boldsymbol{\alpha})\|_{\mathbb{R}^n}^2 &= \sum_j \left| \int_0^R \int_{\Gamma(t)} \nabla_{\bar{g}(t)} \overline{\mathcal{E}}_{R,t}(\beta(\boldsymbol{\alpha}\cdot\boldsymbol{b}(t)))\nabla_{\bar{g}(t)} E_R(t)(J_0^t\phi_t b_j)\right|^2 \\
&\leq \sum_j \left\|\nabla_{\bar{g}(t)} \overline{\mathcal{E}}_{R,t}(\beta(\boldsymbol{\alpha}\cdot\boldsymbol{b}(t)))\right\|_{L^2(\mathcal{C}_R(t))}^2 \left\|\nabla_{\bar{g}(t)} E_R(t)(J_0^t\phi_t b_j)\right\|_{L^2(\mathcal{C}_R(t))}^2 \\
&\leq C_1 \sum_j \|\beta(\boldsymbol{\alpha}\cdot\boldsymbol{b}(t))\|_{W^{1/2,2}(\Gamma(t))}^2 \left\|\nabla_{\bar{g}(t)} E_R(t)(J_0^t\phi_t b_j)\right\|_{L^2(\mathcal{C}_R(t))}^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 4.3.7)} \\
&\leq C_1 \left\|\beta'\right\|_\infty^2 \sum_j \|\boldsymbol{\alpha}\cdot\boldsymbol{b}(t)\|_{W^{1/2,2}(\Gamma(t))}^2 \left\|\nabla_{\bar{g}(t)} E_R(t)(J_0^t\phi_t b_j)\right\|_{L^2(\mathcal{C}_R(t))}^2 \\
&\leq C_1 \left\|\beta'\right\|_\infty^2 \|\boldsymbol{\alpha}\|_{\mathbb{R}^n}^2 \sum_{i,j} \|b_i(t)\|_{W^{1/2,2}(\Gamma(t))}^2 \left\|\nabla_{\bar{g}(t)} E_R(t)(J_0^t\phi_t b_j)\right\|_{L^2(\mathcal{C}_R(t))}^2
\end{aligned}$$

where $C_1 = C_1(R)$, and this gives

$$\|\mathbf{a}(t,\boldsymbol{\alpha})\|_{\mathbb{R}^n}^2 \leq C_2\,\|\boldsymbol{\alpha}\|_{\mathbb{R}^n}^2 \sum_{i,j} \|\phi_t b_i\|_{W^{1/2,2}(\Gamma(t))}^2 \left\|\nabla_{\bar{g}(t)} E_R(t)(J_0^t\phi_t b_j)\right\|_{L^2(\mathcal{C}_R(t))}^2,$$

so that overall

$$\begin{aligned}
&\|F(t,\boldsymbol{\alpha})\|_{\mathbb{R}^n}^2 \\
&\quad\leq 2\,\|\boldsymbol{\alpha}\|_{\mathbb{R}^n}^2 \left(\|\boldsymbol{W}(t)\|_F^2 + C_2 \sum_{i,j} \|\phi_t b_i\|_{W^{1/2,2}(\Gamma(t))}^2 \left\|\nabla_{\bar{g}(t)} E_R(t)(J_0^t\phi_t b_j)\right\|_{L^2(\mathcal{C}_R(t))}^2 \right)
\end{aligned}$$

and hence

$$\begin{aligned}
&\|F(t,\boldsymbol{\alpha})\|_{\mathbb{R}^n} \\
&\quad\leq C\,\|\boldsymbol{\alpha}\|_{\mathbb{R}^n} \left(\|\boldsymbol{W}(t)\|_F + C_3 \sum_{i,j} \|\phi_t b_i\|_{W^{1/2,2}(\Gamma(t))} \left\|\nabla_{\bar{g}(t)} E_R(t)(J_0^t\phi_t b_j)\right\|_{L^2(\mathcal{C}_R(t))} \right) \\
&\quad\leq 2Cc \left(\|\boldsymbol{W}(t)\|_F + C_3 \sum_{i,j} \|\phi_t b_i\|_{W^{1/2,2}(\Gamma(t))} \left\|\nabla_{\bar{g}(t)} E_R(t)(J_0^t\phi_t b_j)\right\|_{L^2(\mathcal{C}_R(t))} \right).
\end{aligned}$$

The term in the brackets on the right hand side (our $f(t)$) is integrable over $(0,T)$. Now an application of the ODE theory in [130, Problem 30.2] gives the existence of a (global) solution $\tilde{u}_n\colon [0,T] \to V_n(0)$. $\square$

Uniform estimates (in n)

Multiply the Galerkin equation (4.26) by α_j and sum up to get (here we are using an arbitrary extension which is linear)

$$\int_{\Gamma_0} \tilde{u}_n'(t)\tilde{u}_n(t) + \int_{\Gamma_0} \tilde{u}_n(t)^2\phi_{-t}(\nabla_\Gamma \cdot \mathbf{w}) + a_R(t;\phi_t\tilde{u}_n(t), J_0^t\phi_t\tilde{u}_n(t)) = 0.$$

Now, in

$$a_R(t;\phi_t\tilde{u}_n(t), J_0^t\phi_t\tilde{u}_n(t)) = \int_0^R \int_{\Gamma(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}(\beta(\phi_t\tilde{u}_n(t)))\nabla_{\bar{g}(t)}E_R(t)(J_0^t\phi_t\tilde{u}_n(t))$$

let us pick $E_R(t)(J_0^t\phi_t\tilde{u}_n(t)) = J_0^t\beta^{-1}(\overline{\mathcal{E}}_{R,t}(\beta(\phi_t\tilde{u}_n(t))))$, which is valid since

$$\begin{aligned}
&\mathcal{T}_{R,t,y=0}E_R(t)(J_0^t\phi_t\tilde{u}_n(t)) = J_0^t\beta^{-1}(\beta(\phi_t\tilde{u}_n(t))) = J_0^t\phi_t\tilde{u}_n(t)\\
&\mathcal{T}_{R,t,y=R}E_R(t)(J_0^t\phi_t\tilde{u}_n(t)) = 0.
\end{aligned}$$

This gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Gamma_0} \tilde{u}_n(t)^2 + \int_{\Gamma_0} \tilde{u}_n^2(t)\phi_{-t}(\nabla_\Gamma \cdot \mathbf{w})\\
&\quad + \int_{\mathcal{C}_R(t)} J_0^t(\beta^{-1})'(\overline{\mathcal{E}}_{R,t}(\beta(\phi_t\tilde{u}_n(t))))|\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}(\beta(\phi_t\tilde{u}_n(t)))|^2\\
&\quad + \int_{\mathcal{C}_R(t)} \beta^{-1}(\overline{\mathcal{E}}_{R,t}(\beta(\phi_t\tilde{u}_n(t))))\nabla_\Gamma\overline{\mathcal{E}}_{R,t}(\beta(\phi_t\tilde{u}_n(t)))\nabla_\Gamma J_0^t = 0
\end{aligned}$$

and thus

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Gamma_0} \tilde{u}_n(t)^2 + C_1C_{\beta'_{inv}}\int_{\mathcal{C}_R(t)} |\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}(\beta(\phi_t\tilde{u}_n(t)))|^2\\
&\leq \|\nabla_\Gamma \cdot \mathbf{w}\|_\infty \int_{\Gamma_0} \tilde{u}_n(t)^2\\
&+ C_2\int_{\mathcal{C}_R(t)} C_\epsilon|\beta^{-1}(\overline{\mathcal{E}}_{R,t}(\beta(\phi_t\tilde{u}_n(t))))|^2 + \epsilon|\nabla_\Gamma\overline{\mathcal{E}}_{R,t}(\beta(\phi_t\tilde{u}_n(t)))|^2
\end{aligned}$$

wherein we note that

$$\begin{aligned}
\int_0^R \int_{\Gamma(t)} |\beta^{-1}(\overline{\mathcal{E}}_{R,t}(\beta(\phi_t \tilde{u}_n(t))))|^2 &\leq \left\|(\beta^{-1})'\right\|_\infty^2 \int_0^R \int_{\Gamma(t)} |\overline{\mathcal{E}}_{R,t}(\beta(\phi_t \tilde{u}_n(t)))|^2 \\
&\leq C_5(R,\lambda_1) \left\|(\beta^{-1})'\right\|_\infty^2 \|\beta(\phi_t \tilde{u}_n(t))\|_{L^2(\Gamma(t))}^2 \\
&\qquad\qquad\qquad\qquad \text{(by Lemma 4.2.16)} \\
&\leq C_5(R,\lambda_1) \left\|(\beta^{-1})'\right\|_\infty^2 \left\|\beta'\right\|_\infty^2 \|\phi_t \tilde{u}_n(t)\|_{L^2(\Gamma(t))}^2 \\
&\leq C_6(R,\lambda_1) \|\tilde{u}_n(t)\|_{L^2(\Gamma_0)}^2 .
\end{aligned}$$

This implies

$$\max_{t\in[0,T]} \|\tilde{u}_n(t)\|_{L^2(\Gamma_0)} + \left\|\nabla_{\bar{g}} \overline{\mathbb{E}}_R(\beta(\phi_{(\cdot)}\tilde{u}_n))\right\|_{L^2_{L^2(\mathcal{C}_R)}} \leq C$$

for a constant C independent of n.

Remark 4.5.5. We needed to truncate the domain in order to obtain the previous bounds. If the domain was instead the full cylinder $\mathcal{C}(t)$, the extension of the test function would have to include a cut-off function so that it belongs to $L^2_{H^1(\mathcal{C})}$, for example, if ψ_ρ is as in Definition 4.2.7, then we could choose

$$E(t)(J_0^t \phi_t \tilde{u}_n(t)) = J_0^t \beta^{-1}[\mathcal{E}_t(\beta(\phi_t \tilde{u}_n(t)) - \overline{\beta(\phi_t \tilde{u}_n(t))}) + \psi_\rho \overline{\beta(\phi_t \tilde{u}_n(t))}]$$

but this leads to a term of the type

$$\int_0^\infty \int_{\Gamma(t)} \beta^{-1}[\mathcal{E}_t(\beta(\phi_t \tilde{u}_n(t)) - \overline{\beta(\phi_t \tilde{u}_n(t))}) + \psi_\rho \overline{\beta(\phi_t \tilde{u}_n(t))}] \nabla_\Gamma \overline{\mathcal{E}}_t(\beta(\phi_t \tilde{u}_n(t))) \nabla_\Gamma J_0^t$$

and we would have to make restrictive assumptions on the evolution to avoid this term blowing up as we pass to the limit $\rho \to \infty$.

Also, we have, writing $\beta(u_n) = \mathbb{T}_{R,y=0}\overline{\mathbb{E}}_R(\beta(u_n))$ and using the trace inequality and Lemma 4.4.5,

$$\|\beta(u_n)\|_{L^2_{W^{1/2,2}}}^2 \leq C_2 \|\beta(u_n)\|_{L^2_{L^2}}^2 + C_3 \leq C_2 \left\|\beta'\right\|_\infty^2 \|u_n\|_{L^2_{L^2}}^2 + C_3 \leq C_4$$

by the energy estimates. Since β^{-1} is Lipschitz, this implies

$$\|u_n\|_{L^2_{W^{1/2,2}}} \leq C$$

independent of n (using the boundedness result of Lemma 4.5.3). The bound on the time derivative follows too: take $\eta \in W^{1/2,2}(\Gamma_0)$ and consider

$$\begin{aligned}
&\langle \tilde{u}_n'(t), \eta\rangle_0 \\
&\quad = (\tilde{u}_n'(t), P_n^0\eta)_{L^2(\Gamma_0)} \\
&\quad = -\int_{\Gamma_0} \tilde{u}_n(t) P_n^0(\eta)\phi_{-t}(\nabla_\Gamma \cdot \mathbf{w}) \\
&\qquad - \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)} \overline{\mathcal{E}}_{R,t}(\beta(\phi_t \tilde{u}_n(t)))\nabla_{\bar{g}(t)} E_R(t)(J_0^t \phi_t P_n^0(\eta)) \\
&\hspace{20em} \text{(rearranging (4.26), assuming linear extension)} \\
&\quad \leq C_1 \|\tilde{u}_n(t)\|_{L^2(\Gamma_0)} \left\|P_n^0(\eta)\right\|_{L^2(\Gamma_0)} \\
&\qquad + \left\|\nabla_{\bar{g}(t)} \overline{\mathcal{E}}_{R,t}(\beta(\phi_t \tilde{u}_n(t)))\right\|_{L^2(\mathcal{C}_R(t))} \left\|\nabla_{\bar{g}(t)} [J_0^t \overline{\mathcal{E}}_{R,t}(\phi_t P_n^0(\eta))]\right\|_{L^2(\mathcal{C}_R(t))}
\end{aligned}$$

after picking $E_R(t)(J_0^t\phi_t P_n^0(\eta)) = J_0^t \overline{\mathcal{E}}_{R,t}(\phi_t P_n^0(\eta))$. Observe that

$$\begin{aligned}
&\left\|\nabla_{\bar{g}(t)} [J_0^t \overline{\mathcal{E}}_{R,t}(\phi_t P_n^0(\eta))]\right\|_{L^2(\mathcal{C}_R(t))}^2 \\
&\quad \leq C_2 \int_{\mathcal{C}_R(t)} |\nabla_{\bar{g}(t)} \overline{\mathcal{E}}_{R,t}(\phi_t P_n^0(\eta))|^2 + |\overline{\mathcal{E}}_{R,t}(\phi_t P_n^0(\eta))|^2 \\
&\quad \leq C_3(\lambda_1, R) \left\|\phi_t P_n^0(\eta)\right\|_{W^{1/2,2}(\Gamma(t))}^2 \qquad \text{(by Lemmas 4.3.7 and 4.2.16)} \\
&\quad \leq C_4(\lambda_1, R) \left\|P_n^0(\eta)\right\|_{W^{1/2,2}(\Gamma_0)}^2
\end{aligned}$$

which implies

$$\begin{aligned}
\int_0^T \langle \tilde{u}_n'(t), \eta\rangle_0 &\leq C_5 \|\tilde{u}_n\|_{L^2(0,T;L^2(\Gamma_0))} \left\|P_n^0(\eta)\right\|_{L^2(0,T;L^2(\Gamma_0))} \\
&\quad + \left\|\nabla_{\bar{g}(t)} \overline{\mathbb{E}}_R(\beta(\phi_t \tilde{u}_n))\right\|_{L^2_{L^2(\mathcal{C}_R)}} \left\|P_n^0(\eta)\right\|_{L^2(0,T;W^{1/2,2}(\Gamma_0))} \\
&\leq C_7 \|\eta\|_{L^2(0,T;W^{1/2,2}(\Gamma_0))}
\end{aligned}$$

by using the uniform estimates. Taking the supremum over $\eta \in L^2(0,T;W^{1/2,2}(\Gamma_0))$ shows that

$$\left\|\tilde{u}_n'\right\|_{L^2(0,T;W^{-1/2,2}(\Gamma_0))} \leq C.$$

Passage to the limit in the Galerkin approximation

We have as $n \to \infty$

$$\begin{aligned}
\tilde{u}_n &\rightharpoonup \tilde{u} \quad \text{in } L^2(0,T;W^{1/2,2}(\Gamma_0)) \\
\tilde{u}_n' &\rightharpoonup \tilde{u}' \quad \text{in } L^2(0,T;W^{-1/2,2}(\Gamma_0)) \\
\tilde{u}_n &\to \tilde{u} \quad \text{in } L^2(0,T;L^2(\Gamma_0)) \\
\underline{D}_i \overline{\mathbb{E}}_R \beta(u_n) &\rightharpoonup \theta_i \quad \text{in } L^2_{L^2(\mathcal{C}_R)} \\
\partial_y \overline{\mathbb{E}}_R \beta(u_n) &\rightharpoonup \theta_y \quad \text{in } L^2_{L^2(\mathcal{C}_R)}
\end{aligned} \tag{4.28}$$

where $\underline{D}_i = (\nabla_\Gamma)_i$ is the i-th component of the tangential gradient and Aubin–Lions yielded the strong convergence. Therefore, we have $u_n \to u$ in $L^2_{L^2}$ and $\beta(u_n) \to \beta(u)$ in $L^2_{L^2}$ thanks to the Lipschitz continuity of β. Due to the boundedness result in Lemma 4.2.16 on $\overline{\mathcal{E}}_{R,t}$ with the constants in the bound depending on $\lambda_1(t)$ (which can be bounded), we see that

$$\overline{\mathbb{E}}_R(\beta(u_n)) \to \overline{\mathbb{E}}_R(\beta(u)) \quad \text{in } L^2_{L^2(\mathcal{C}_R)}. \tag{4.29}$$

Identification of the spatial term Take the test function $\eta \in L^2_{H^1(\mathcal{C}_R)}$ defined by

$$\eta(t,y,x) = \psi(t)\phi_t v_0 h(y) \quad \text{where } \psi \in C_c^\infty(0,T),\ v_0 \in C_c^1(\Gamma_0) \text{ and } h \in C_c^\infty(0,R). \tag{4.30}$$

Consider the spatial integration by parts formula (see Chapter 2) on $\Gamma(t)$ integrated over y and t:

$$\int_0^T \int_{\mathcal{C}_R(t)} (\underline{D}_i \overline{\mathbb{E}}_R \beta(u_n))\eta = -\int_0^T \int_{\mathcal{C}_R(t)} (\overline{\mathbb{E}}_R \beta(u_n))\underline{D}_i \eta + \int_0^T \int_{\mathcal{C}_R(t)} (\overline{\mathbb{E}}_R \beta(u_n))\eta H \nu_i^\Gamma$$

Using (4.28) on the left hand side and (4.29) on the right hand side, we have

$$\int_0^T \int_{\mathcal{C}_R(t)} \theta_i \eta = -\int_0^T \int_{\mathcal{C}_R(t)} (\overline{\mathbb{E}}_R \beta(u))\underline{D}_i \eta + \int_0^T \int_{\mathcal{C}_R(t)} (\overline{\mathbb{E}}_R \beta(u))\eta H \nu_i^\Gamma.$$

It follows that for almost every t, for almost every y,

$$\int_{\Gamma(t)} \theta_i(t,y)\phi_t v_0 = -\int_{\Gamma(t)} (\overline{\mathbb{E}}_R \beta(u))(t,y)\underline{D}_i \phi_t v_0 + \int_{\Gamma(t)} (\overline{\mathbb{E}}_R \beta(u))(t,y)\phi_t v_0 H \nu_i^\Gamma,$$

and since this holds for all $\phi_t v_0 \in C_c^1(\Gamma_0)$, it also holds for all $v \in C_c^1(\Gamma(t))$. This implies that $\underline{D}_i(\overline{\mathbb{E}}_R \beta(u)) = \theta_i(t,y)$ by definition.

Identification of the y term Again take $\eta \in L^2_{H^1(\mathcal{C}_R)}$ as in (4.30) and consider the integration by parts formula

$$\int_0^T \int_{\mathcal{C}_R(t)} (\partial_y \overline{\mathbb{E}}_R \beta(u_n))\eta = -\int_0^T \int_{\mathcal{C}_R(t)} (\overline{\mathbb{E}}_R \beta(u_n))\partial_y \eta. \tag{4.31}$$

This formula holds because

$$\begin{aligned}\int_0^T \int_{\mathcal{C}_R(t)} (\partial_y \overline{\mathbb{E}}_R \beta(u_n))\eta &= \int_0^T \psi(t) \int_0^R h(y) \int_{\Gamma(t)} \partial_y \overline{\mathbb{E}}_R \beta(u_n)\phi_t v_0 \\ &= \int_0^T \psi(t) \int_0^R h(y) \frac{d}{dy} \int_{\Gamma(t)} \overline{\mathbb{E}}_R \beta(u_n)\phi_t v_0\end{aligned}$$

(because $\overline{\mathbb{E}}_R \beta(u_n) \in H^1(\mathcal{C}_R(t))$ so the inner product over $\Gamma(t)$ is absolutely continuous)

$$= -\int_0^T \psi(t) \int_0^R \partial_y h(y) \int_{\Gamma(t)} \overline{\mathbb{E}}_R \beta(u_n)\phi_t v_0.$$

Passing to the limit in (4.31), we find

$$\int_0^T \int_{\mathcal{C}_R(t)} \theta_y \eta = -\int_0^T \int_{\mathcal{C}_R(t)} (\overline{\mathbb{E}}_R \beta(u))\partial_y \eta.$$

It follows that for almost every t

$$\int_{\Gamma(t)} \phi_t v_0 \int_0^R \theta_y h(y) = -\int_{\Gamma(t)} \phi_t v_0 \int_0^R (\overline{\mathbb{E}}_R \beta(u))\partial_y h(y).$$

which holds for all $C^1_c(\Gamma)$, implying for almost all t and almost all x that

$$\int_0^R \theta_y(t,x)h(y) = -\int_0^R (\overline{\mathbb{E}}_R \beta(u))(t,x)\partial_y h(y)$$

and thus we identify $\theta_y = \partial_y \overline{\mathbb{E}}_R \beta(u)$.

Conclusions Therefore, we get

$$\begin{aligned}\tilde{u}_n &\rightharpoonup \tilde{u} && \text{in } L^2(0,T;W^{1/2,2}(\Gamma_0))\\ \tilde{u}_n' &\rightharpoonup \tilde{u}' && \text{in } L^2(0,T;W^{-1/2,2}(\Gamma_0))\\ \tilde{u}_n &\to \tilde{u} && \text{in } L^2(0,T;L^2(\Gamma_0))\\ \nabla_{\bar{g}} \overline{\mathbb{E}}_R \beta(u_n) &\rightharpoonup \nabla_{\bar{g}} \overline{\mathbb{E}}_R \beta(u) && \text{in } L^2_{L^2(\mathcal{C}_R)}\end{aligned}$$

Recall the Galerkin equation (4.26) and recall that $V_n(t) := \text{span}\{\phi_t b_1, ..., \phi_t b_n\}$. Given $\eta \in L^2_{W^{1/2,2}}$, by density, there is a sequence $\{\eta_l\}$ with $\eta_l \in L^2_{V_l}$ for each l such that $\eta_l \to \eta$ in $L^2_{W^{1/2,2}}$ and

$$\eta_l(t) = \sum_{j=1}^{l} \gamma_j^l(t)\phi_t b_j.$$

If $l \leq n$, then $\eta_l \in L^2_{V_n}$ and we multiply the above by $\gamma_j^l(t)$ and sum up to get

$$\int_{\Gamma_0} \tilde{u}_n'(t)\tilde{\eta}_l(t) + \int_{\Gamma_0} \tilde{u}_n(t)\tilde{\eta}_l(t)\phi_{-t}(\nabla_\Gamma \cdot \mathbf{w}) \\ + \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}(\beta(u_n(t)))\nabla_{\bar{g}(t)}E_R(t)(J_0^t\eta_l(t)) = 0$$

where $\tilde{\eta}_l(t) = \phi_{-t}\eta_l(t)$. We obtain after integrating the above equation and taking the limit as $n \to \infty$ the equation

$$\int_0^T \langle \tilde{u}'(t), \tilde{\eta}_l(t)\rangle_0 + \int_0^T \int_{\Gamma_0} \tilde{u}(t)\tilde{\eta}_l(t)\phi_{-t}(\nabla_\Gamma \cdot \mathbf{w}) \\ + \int_0^T \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}(\beta(u(t)))\nabla_{\bar{g}(t)}E_R(t)(J_0^t\eta_l(t)) = 0. \tag{4.32}$$

Let us prove that $\phi_{-t}(J_0^t)\tilde{\eta}_l \to \phi_{-t}(J_0^t)\tilde{\eta}$ in $L^2(0,T;W^{1/2,2}(\Gamma_0))$. The L^2 convergence is obvious, and for the seminorm, we have

$$\begin{aligned}
&\left|\phi_{-t}(J_0^t)(\tilde{\eta}_l(t) - \tilde{\eta}(t))\right|^2_{W^{1/2,2}(\Gamma_0)} \\
&= \int_{\Gamma_0}\int_{\Gamma_0} \frac{|[\phi_{-t}(J_0^t)(\tilde{\eta}_l(t) - \tilde{\eta}(t))](x) - [\phi_{-t}(J_0^t)(\tilde{\eta}_l(t) - \tilde{\eta}(t))](y)|^2}{|x-y|^n} \\
&\leq 2\int_{\Gamma_0}\int_{\Gamma_0} \frac{|\phi_{-t}(J_0^t)(x)\left([\tilde{\eta}_l(t) - \tilde{\eta}(t)](x) - [\tilde{\eta}_l(t) - \tilde{\eta}(t)](y)\right)|^2}{|x-y|^n} \\
&\quad + 2\int_{\Gamma_0}\int_{\Gamma_0} \frac{|[\tilde{\eta}_l(t) - \tilde{\eta}(t)](y)(\phi_{-t}(J_0^t)(x) - \phi_{-t}(J_0^t)(y)|^2}{|x-y|^n} \\
&\leq C_1\int_{\Gamma_0}\int_{\Gamma_0} \frac{|[\tilde{\eta}_l(t) - \tilde{\eta}(t)](x) - [\tilde{\eta}_l(t) - \tilde{\eta}(t)](y)|^2}{|x-y|^n} + C_2\int_{\Gamma_0}\int_{\Gamma_0} \frac{|[\tilde{\eta}_l(t) - \tilde{\eta}(t)](y)|^2}{|x-y|^{n-2}} \\
&\qquad\qquad (\phi_{-t}J_0^t \text{ is Lipschitz with uniform Lipschitz constant}) \\
&= C_1\left|\tilde{\eta}_l(t) - \tilde{\eta}(t)\right|^2_{W^{1/2,2}(\Gamma_0)} + C_2\int_{\Gamma_0} |[\tilde{\eta}_l(t) - \tilde{\eta}(t)](y)|^2 \int_{\Gamma_0} \frac{1}{|x-y|^{n-2}}\,\mathrm{d}x\mathrm{d}y \\
&\leq C_1\left|\tilde{\eta}_l(t) - \tilde{\eta}(t)\right|^2_{W^{1/2,2}(\Gamma_0)} + C_3\int_{\Gamma_0} |[\tilde{\eta}_l(t) - \tilde{\eta}(t)](y)|^2 \\
&\qquad\qquad (\text{for example, see Lemma 2.5.2}) \\
&\leq C_4\left\|\tilde{\eta}_l(t) - \tilde{\eta}(t)\right\|^2_{W^{1/2,2}(\Gamma_0)}.
\end{aligned}$$

Integrating over time and passing to the limit shows the result. Thus $J_0^t\eta_l \to J_0^t\eta$ in $L^2_{W^{1/2,2}}$ and it follows from Lemma 4.4.8 that $\nabla_{\bar{g}}\overline{\mathbb{E}}_R(\eta_l J_0^t) \to \nabla_{\bar{g}}\overline{\mathbb{E}}_R(\eta J_0^t)$ in $L^2_{L^2(\mathcal{C}_R)}$. With this in mind, taking limits $l \to \infty$ in (4.32) with $E_R = \overline{\mathbb{E}}_R$ yields

$$\int_0^T \langle \tilde{u}'(t), \tilde{\eta}(t)\rangle_0 + \int_0^T \int_{\Gamma_0} \tilde{u}(t)\tilde{\eta}(t)\phi_{-t}(\nabla_\Gamma \cdot \mathbf{w})$$
$$+ \int_0^T \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}(\beta(u(t)))\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}(J_0^t\eta(t)) = 0,$$

but because the extension can be arbitrary (by Remark 4.2.18), we get

$$\int_0^T \langle \tilde{u}'(t), \tilde{\eta}(t)\rangle_0 + \int_0^T \int_{\Gamma_0} \tilde{u}(t)\tilde{\eta}(t)\phi_{-t}(\nabla_\Gamma \cdot \mathbf{w})$$
$$+ \int_0^T \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}(\beta(u(t)))\nabla_{\bar{g}(t)}E_R(t)(J_0^t\eta(t)) = 0$$

for all $\tilde{\eta} \in L^2(0,T;W^{1/2,2}(\Gamma_0))$. Now, pushing forward the integrals, recalling from the proof of Theorem 1.2.46 that $\dot{u}(t) = \phi_{-t}^*(J_t^0\tilde{u}'(t))$ and using

$$\begin{aligned}\langle \tilde{u}'(t), \tilde{\eta}(t)\rangle_0 &= \langle J_t^0\tilde{u}'(t), (J_t^0)^{-1}\tilde{\eta}(t)\rangle_0\\ &= \langle \phi_{-t}^*(J_t^0\tilde{u}'(t)), \phi_t((J_t^0)^{-1}\tilde{\eta}(t))\rangle\\ &= \langle \dot{u}(t), J_0^t\phi_t\tilde{\eta}(t)\rangle\end{aligned}$$

gives

$$\int_0^T \langle \dot{u}(t), J_0^t\phi_t\tilde{\eta}(t)\rangle + \int_0^T \int_{\Gamma(t)} u(t)\phi_t\tilde{\eta}(t)J_0^t\nabla_\Gamma \cdot \mathbf{w}$$
$$+ \int_0^T \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}(\beta(u(t)))\nabla_{\bar{g}(t)}E_R(t)(J_0^t\phi_t\tilde{\eta}(t)) = 0.$$

Finally, picking $\tilde{\eta} = \phi_{-t}\eta/\phi_{-t}J_0^t$ yields

$$\int_0^T \langle \dot{u}(t), \eta(t)\rangle + \int_0^T \int_{\Gamma(t)} u(t)\eta(t)\nabla_\Gamma \cdot \mathbf{w}$$
$$+ \int_0^T \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}(\beta(u(t)))\nabla_{\bar{g}(t)}E_R(t)\eta(t) = 0$$

for each $\eta \in L^2_{W^{1/2,2}}$. The initial condition can be checked like done in §1.6. This concludes the proof of Theorem 4.5.1.

4.5.2 Existence of solutions to the non-degenerate problem

By the previous subsection, we have a solution $u_R \in \mathbb{W}(W^{1/2,2}, W^{-1/2,2})$ with $u_R(0) = u_0$ and $\nabla_{\bar{g}}\overline{\mathbb{E}}_R(\beta(u_R)) \in L^2_{L^2(\mathcal{C}_R)}$ satisfying

$$\langle \dot{u}_R(t), \eta(t)\rangle + \int_{\Gamma(t)} u_R(t)\eta(t)\nabla_\Gamma \cdot \mathbf{w} + \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}(\beta(u_R(t)))\nabla_{\bar{g}(t)}E_R(t)\eta(t) = 0$$

for all $\eta \in L^2_{W^{1/2,2}}$ and for almost all t.

Uniform estimates (in R)

We now want to obtain estimates independent of R. Let us pick $\eta = u_R$ and use $E_R(\eta) = \beta^{-1}(\overline{\mathbb{E}}_R(\beta(u_R)))$ which equals u_R at $y = 0$ and 0 at $y = R$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)} |u_R(t)|^2 + \frac{1}{2}\int_{\Gamma(t)} |u_R(t)|^2 \nabla_\Gamma \cdot \mathbf{w}&\\ + \int_{\mathcal{C}_R(t)} (\beta^{-1})'[\overline{\mathcal{E}}_{R,t}(\beta(u_R(t)))]|\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}(\beta(u_R(t)))|^2 &= 0.\end{aligned}$$

Since $(\beta^{-1})' \geq C_{\beta'_{inv}}$ and $u_R(0) = u_0$, we immediately obtain via Gronwall's inequality that

$$\max_{t\in[0,T]} \|u_R(t)\|_{L^2(\Gamma(t))} + \left\|\nabla_{\bar{g}}\overline{\mathbb{E}}_R(\beta(u_R))\right\|_{L^2_{L^2(C_R)}} \leq C$$

independent of R. Rearranging the equation gives

$$\begin{aligned}&\int_0^T \langle \dot{u}_R(t), \eta(t)\rangle_{W^{-1/2,2}(\Gamma(t)),W^{1/2,2}(\Gamma(t))}\\
&= -\int_0^T\int_{\Gamma(t)} u_R(t)\eta(t)\nabla_\Gamma \cdot \mathbf{w} - \int_0^T\int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_{R,t}(\beta(u_R(t)))\nabla_{\bar{g}(t)}E_R(t)\eta(t)\\
&\leq C\|u_R\|_{L^2_{L^2}}\|\eta\|_{L^2_{L^2}} + \left\|\nabla_{\bar{g}}\overline{\mathbb{E}}_R(\beta(u_R))\right\|_{L^2_{L^2(\mathcal{C}_R)}}\left\|\nabla_{\bar{g}}\overline{\mathbb{E}}_R\eta\right\|_{L^2_{L^2(\mathcal{C}_R)}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(picking } E_R = \overline{\mathbb{E}}_R\text{)}\\
&\leq C_1\|\eta\|_{L^2_{L^2}} + C_2\|\eta\|_{L^2_{W^{1/2,2}}}\end{aligned}$$

using Lemma 4.4.8 on the rightmost term and by the uniform estimates. Taking supremums over $\eta \in L^2_{W^{1/2,2}}$ gives

$$\|\dot{u}_R\|_{L^2_{W^{-1/2,2}}} \leq C.$$

Lemma 4.2.12 implies that

$$|\mathcal{T}_t v|_{W^{1/2,2}(\Gamma(t))} \leq C \left\|\nabla_{\bar{g}(t)} v\right\|_{L^2(\mathcal{C}(t))} \qquad \text{for all } v \in H^1(\mathcal{C}(t)),$$

where we now claim that the constant is independent of t. Indeed, an inspection of the proof reveals that we need to check whether the trace map $\mathcal{T}_t \colon H^1(\mathcal{C}(t)) \to W^{1/2,2}(\Gamma(t))$ is bounded uniformly in t and whether the constant in the Poincaré inequality on $\Gamma(t)$ is independent of t. The first question has been settled by (4.17) and the second is also affirmative due to (2.25). Using this inequality, we find

$$\begin{aligned}
|\beta(u_R(t))|_{W^{1/2,2}(\Gamma(t))} &= \left|\mathcal{T}_t \mathscr{Z}_R \overline{\mathcal{E}}_{R,t}\beta(u_R(t))\right|_{W^{1/2,2}(\Gamma(t))} \\
&\leq C \left\|\nabla_{\bar{g}} \mathscr{Z}_R \overline{\mathcal{E}}_{R,t}(\beta(u_R(t)))\right\|_{L^2(\mathcal{C}(t))} \\
&= C \left\|\nabla_{\bar{g}} \overline{\mathcal{E}}_{R,t}(\beta(u_R(t)))\right\|_{L^2(\mathcal{C}_R(t))}
\end{aligned}$$

which implies that $\int_0^T |\beta(u_R(t))|^2_{W^{1/2,2}(\Gamma(t))} \leq C$. This gives boundedness of u_R in the fractional seminorm, and thus we have that

$$\|u_R\|_{L^2_{W^{1/2,2}}} \leq C.$$

Passage to the limit in R

So we have

$$\begin{aligned}
u_R &\rightharpoonup u && \text{in } L^2_{W^{1/2,2}} \\
\dot{u}_R &\rightharpoonup \dot{u} && \text{in } L^2_{W^{-1/2,2}} \\
u_R &\to u && \text{in } L^2_{L^2} \\
\underline{D}_i \mathscr{Z}_R \overline{\mathbb{E}}_R(\beta(u_R)) &\rightharpoonup \theta_i && \text{in } L^2_{L^2(C)} \\
\partial_y \mathscr{Z}_R \overline{\mathbb{E}}_R(\beta(u_R)) &\rightharpoonup \theta_y && \text{in } L^2_{L^2(C)},
\end{aligned}$$

and we need to identify the limits. Our first task is to show that $\mathscr{Z}_R \overline{\mathbb{E}}_R(\beta(u_R) - \overline{\beta(u_R)}) \to \mathbb{E}(\beta(u) - \overline{\beta(u)})$ in $L^2_{L^2(C)}$.

Set $w_R = \beta(u_R)$ and $w = \beta(u)$; since $w_R(t) - \bar{w}_R(t) \to w(t) - \bar{w}(t)$ in $L^2(\Gamma(t))$ for a.e t, Lemma 4.2.22 implies

$$\mathscr{Z}_R \mathcal{E}_{R,t}(w_R(t) - \bar{w}_R(t)) \to \mathcal{E}_t(w(t) - \bar{w}(t)) \quad \text{in } L^2(\mathcal{C}(t)).$$

So we have for a.e. t

$$f_R(t) := \|\mathscr{Z}_R \mathcal{E}_R(w_R(t) - \bar{w}_R(t)) - \mathcal{E}(w(t) - \bar{w}(t))\|^2_{L^2(\mathcal{C}(t))} \to 0.$$

Note that

$$\begin{aligned}
|f_R(t)| &\le 2\left\|\mathscr{Z}_R\mathcal{E}_R(w_R(t)-\bar{w}_R(t))\right\|^2_{L^2(\mathcal{C}(t))} + 2\left\|\mathcal{E}(w(t)-\bar{w}(t))\right\|^2_{L^2(\mathcal{C}(t))}\\
&= 2\left\|\mathcal{E}_R(w_R(t)-\bar{w}_R(t))\right\|^2_{L^2(\mathcal{C}_R(t))} + 2\left\|\mathcal{E}(w(t)-\bar{w}(t))\right\|^2_{L^2(\mathcal{C}(t))}\\
&\le C_1\left(\left\|w_R(t)-\bar{w}_R(t)\right\|^2_{L^2(\Gamma(t))} + \left\|w(t)-\bar{w}(t)\right\|^2_{L^2(\Gamma(t))}\right) =: g_R(t)
\end{aligned}$$

by Lemma 4.2.16 and (4.11). Now for a.a. t,

$$g_R(t) \to 2C_1\left\|w(t)-\bar{w}(t)\right\|^2_{L^2(\Gamma(t))}$$

and also

$$\int_0^T g_R(t) = C_1(\|w_R - \bar{w}_R\|^2_{L^2_{L^2}} + \|w-\bar{w}\|^2_{L^2_{L^2}}) \to 2C_1\left\|w-\bar{w}\right\|^2_{L^2_{L^2}}$$

so that by the generalised DCT, $\int_0^T f_R(t) \to \int_0^T \lim_{R\to\infty} f_R(t) = 0$, giving

$$\mathscr{Z}_R\mathbb{E}_R(w_R - \bar{w}_R) \to \mathbb{E}(w-\bar{w}) \quad \text{in } L^2_{L^2(\mathcal{C})}$$

as desired. Now, choosing η as in (4.30) except with $h \in C_c^\infty(0,\infty)$, we have

$$\begin{aligned}
\int_0^T\int_{\mathcal{C}(t)} \underline{D}_i\mathscr{Z}_R\overline{\mathbb{E}}_R w_R\eta &= \int_0^T\int_{\mathcal{C}(t)} \underline{D}_i\mathscr{Z}_R\left(\mathbb{E}_R(w_R-\bar{w}_R) + \frac{R-y}{R}\bar{w}_R\right)\eta\\
&= \int_0^T\int_{\mathcal{C}(t)} \underline{D}_i\mathscr{Z}_R\mathbb{E}_R(w_R-\bar{w}_R)\eta\\
&= -\int_0^T\int_{\mathcal{C}(t)} \underline{D}_i\eta\mathscr{Z}_R\mathbb{E}_R(w_R-\bar{w}_R)\\
&\quad -\int_0^T\int_{\mathcal{C}(t)} \mathscr{Z}_R\mathbb{E}_R(w_R-\bar{w}_R)\eta H\nu_i^\Gamma,
\end{aligned}$$

and passing to the limit on both sides gives

$$-\int_0^T\int_{\mathcal{C}(t)} \underline{D}_i\eta\mathbb{E}(w-\bar{w}) + \int_0^T\int_{\mathcal{C}(t)} \mathbb{E}(w-\bar{w})\eta H\nu_i^\Gamma = \int_0^T\int_{\mathcal{C}(t)} \theta_i\eta,$$

and then an argument similar to that in §4.5.1 shows that $\underline{D}_i\mathbb{E}(w-\bar{w}) = \theta_i$. For the y term,

$$\begin{aligned}\int_0^T \int_{\mathcal{C}(t)} \partial_y \mathscr{L}_R \overline{\mathbb{E}}_R w_R \eta &= \int_0^T \int_{\mathcal{C}(t)} \partial_y \mathscr{L}_R \left(\mathbb{E}_R(w_R - \bar{w}_R) + \frac{R-y}{R}\bar{w}_R \right) \eta \\ &= \int_0^T \int_{\mathcal{C}(t)} (\partial_y \mathscr{L}_R \mathbb{E}_R(w_R - \bar{w}_R) - \chi_{y\leq R}\frac{\bar{w}_R}{R})\eta \\ &= -\int_0^T \int_{\mathcal{C}(t)} \mathscr{L}_R \mathbb{E}_R(w_R - \bar{w}_R)\partial_y \eta + \chi_{y\leq R}\frac{\bar{w}_R}{R}\eta, \end{aligned} \tag{4.33}$$

and the last term on the right hand side

$$\int_0^T \int_{\mathcal{C}(t)} \chi_{y\leq R}\frac{\bar{w}_R}{R}\eta = \int_0^T \frac{\bar{w}_R(t)}{R}\psi(t) \int_0^\infty \chi_{y\leq R}(y)h(y) \int_{\Gamma(t)} \phi_t v_0 \to 0$$

since $\int_0^T \bar{w}_R(t)/R\psi(t) \to 0$ and $\int_0^\infty \chi_{y\leq R}(y)h(y) \to \int_0^\infty h(y)$ both due to the DCT (recall that $\bar{w}_R(t) \to \bar{w}(t)$ a.e.). Then taking the limit on the LHS and the RHS of (4.33), we get

$$\int_0^T \int_{\mathcal{C}(t)} \theta_y \eta = -\int_0^T \int_{\mathcal{C}(t)} \mathbb{E}(w - \bar{w})\partial_y \eta$$

which again gives $\partial_y \mathbb{E}(w - \bar{w}) = \theta_y$ by similar reasoning to §4.5.1. Now we may pass to the limit in the equation

$$\begin{aligned}\int_0^T \langle \dot{u}_R(t), \eta(t)\rangle &+ \int_0^T \int_{\Gamma(t)} u_R(t)\eta(t)\nabla_\Gamma \cdot \mathbf{w} \\ &+ \int_0^T \int_{\mathcal{C}_R(t)} \nabla_{\bar{g}(t)} \overline{\mathcal{E}}_{R,t}(\beta(u_R(t)))\nabla_{\bar{g}(t)} E_R(t)\eta(t) = 0.\end{aligned}$$

First of all, take $E_R\eta = \mathscr{L}_1\overline{\mathbb{E}}_1\eta$ (this satisfies $E_R\eta|_{y=0} = \eta$ and $E_R\eta|_{y=R} = 0$), then replace the integral over $(0, R)$ by one over $(0, \infty)$:

$$\begin{aligned}\int_0^T \langle \dot{u}_R(t), \eta(t)\rangle &+ \int_0^T \int_{\Gamma(t)} u_R(t)\eta(t)\nabla_\Gamma \cdot \mathbf{w} \\ &+ \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} \mathscr{L}_R \overline{\mathcal{E}}_{R,t}(\beta(u_R(t)))\nabla_{\bar{g}(t)} \mathscr{L}_1\overline{\mathbb{E}}_1\eta(t) = 0\end{aligned}$$

and then use the above convergence results:

$$\int_0^T \langle \dot{u}(t), \eta(t)\rangle + \int_0^T \int_{\Gamma(t)} u(t)\eta(t)\nabla_\Gamma \cdot \mathbf{w} \\ + \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} \overline{\mathcal{E}}_t(\beta(u(t)))\nabla_{\bar{g}(t)} \mathscr{L}_1 \mathbb{E}_1 \eta(t) = 0$$

and then recall that the elliptic form can have an arbitary extension, so that the above equals

$$\int_0^T \langle \dot{u}(t), \eta(t)\rangle + \int_0^T \int_{\Gamma(t)} u(t)\eta(t)\nabla_\Gamma \cdot \mathbf{w} \\ + \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} \overline{\mathcal{E}}_t(\beta(u(t)))\nabla_{\bar{g}(t)} (E\eta)(t) = 0$$

for any extension $E\eta \in L^2_{H^1(\mathcal{C})}$ which has trace at $y = 0$ equal to η. For the conservation of mass, note that $\int_{\Gamma(t)} u_R(t) = \int_{\Gamma_0} u_0$ holds simply by testing with $\eta = E_R\eta \equiv 1$. By the strong convergence of u_R to u in $L^2_{L^2}$, $\int_{\Gamma(t)} u(t) = \int_{\Gamma_0} u_0$ follows for almost all t, but since the left hand side is a continuous function of t, it holds for all t.

4.5.3 Contraction principle

This part is similar to what is done in [15]. Let u_0^1, $u_0^2 \in L^\infty(\Gamma_0)$ be initial data and consider the respective solutions u_{1R} and u_{2R} to the truncated problem (4.9). The difference of the solutions satisfies

$$\langle \dot{u}_{1R}(t) - \dot{u}_{2R}(t), \eta(t)\rangle + \int_{\Gamma(t)} (u_{1R}(t) - u_{2R}(t))\eta(t)\nabla_\Gamma \cdot \mathbf{w} \\ + \int_0^R \int_{\Gamma(t)} \nabla_{\bar{g}(t)} (\overline{\mathcal{E}}_{R,t}(\beta(u_{1R}(t))) - \overline{\mathcal{E}}_{R,t}(\beta(u_{2R}(t))))\nabla_{\bar{g}(t)} E_R(t)\eta(t) = 0.$$

Define $F_\epsilon(r) = \frac{1}{\epsilon}T_\epsilon(r^+)$ and pick $\eta = F_\epsilon(u_{1R} - u_{2R})$. Set $v_{1R} = \overline{\mathbb{E}}_R(\beta(u_{1R}))$, and let us pick $E_R\eta = F_\epsilon[\beta^{-1}(v_{1R}) - \beta^{-1}(v_{2R})]$ which satisfies $\mathbb{T}_{R,y=0}(E_R\eta) = F_\epsilon(u_{1R} - u_{2R}) = \eta$ and $\mathbb{T}_{R,y=R}(E_R\eta) = \frac{1}{\epsilon}T_\epsilon(0) = 0$, and $E_R\eta \in L^2_{H^1(\mathcal{C}_R)}$, so is an admissible test function. Note that here we used that, for example, $\mathcal{T}_{R,y=0}F_\epsilon(w) = F_\epsilon(\mathcal{T}_{R,y=0}w)$ for all $w \in H^1(\mathcal{C}_R(0))$. This holds for all smooth functions dense in $H^1(\mathcal{C}_R(0))$, then we can simply use the continuity of $F_\epsilon(\cdot) = \frac{1}{\epsilon}T_\epsilon \circ (\cdot)^+ \colon H^1(\mathcal{C}_R(0)) \to H^1(\mathcal{C}_R(0))$ (see §4.3.3) and of $F_\epsilon \colon W^{1/2,2}(\Gamma_0) \to W^{1/2,2}(\Gamma_0)$ (by Lemma 4.5.3, since F_ϵ is Lipschitz). We also used that $\mathcal{T}_{R,y=0}\beta^{-1}(w) = \beta^{-1}(\mathcal{T}_{R,y=0}w)$ for all $w \in H^1(\mathcal{C}_R(0))$, which again follows again by Lemma 4.5.3 and the continuity of $\beta^{-1}\colon H^1(\mathcal{C}_R(0)) \to H^1(\mathcal{C}_R(0))$ (which follows from the boundedness and continuity of $(\beta^{-1})'$).

Testing with this and integrating over time:

$$\int_0^t \langle \partial^\bullet(u_{1R} - u_{2R}), F_\epsilon(u_{1R} - u_{2R})\rangle + \int_0^t \int_{\Gamma_0} (u_{1R} - u_{2R}) F_\epsilon(u_{1R} - u_{2R}) \nabla_\Gamma \cdot \mathbf{w}$$
$$+ \int_0^t \int_{\mathcal{C}(s)} \nabla_{\bar{g}(s)}(v_{1R}(s) - v_{2R}(s)) \nabla_{\bar{g}(s)} E_R(s)\eta(s) = 0. \tag{4.34}$$

Now define $S_\epsilon(s) = \int_0^s F_\epsilon(r)\,\mathrm{d}r$. By Lemma 4.3.12,

$$\int_0^t \langle \partial^\bullet(u_{1R} - u_{2R}), F_\epsilon(u_{1R} - u_{2R})\rangle = \int_{\Gamma(t)} S_\epsilon(u_{1R}(t) - u_{2R}(t))$$
$$- \int_{\Gamma_0} S_\epsilon(u_{1R}(0) - u_{2R}(0)) - \int_0^t \int_{\Gamma(s)} S_\epsilon(u_{1R}(s) - u_{2R}(s)) \nabla_\Gamma \cdot \mathbf{w},$$

and using that $S_\epsilon(\cdot) \to (\cdot)^+$ pointwise as $\epsilon \to 0$ and $|S_\epsilon(r)| \leq \frac{3}{2} + |r|$, we obtain by DCT

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \langle \partial^\bullet(u_{1R}(s) - u_{2R}(s)), T_\epsilon(u_{1R}(s) - u_{2R}(s)))\rangle$$
$$= \int_{\Gamma(t)} (u_{1R}(t) - u_{2R}(t))^+ - \int_{\Gamma_0} (u_{1R}(0) - u_{2R}(0))^+$$
$$- \int_0^t \int_{\Gamma(s)} (u_{1R}(s) - u_{2R}(s))^+ \nabla_\Gamma \cdot \mathbf{w}. \tag{4.35}$$

Taking the lim inf on equation (4.34), bearing in mind[3]

$$\int_0^t \int_{\Gamma(s)} (u_{1R} - u_{2R}) F_\epsilon(u_{1R} - u_{2R}) \nabla_\Gamma \cdot \mathbf{w} \to \int_0^t \int_{\Gamma(s)} (u_{1R} - u_{2R})^+ \nabla_\Gamma \cdot \mathbf{w}$$

and using (4.35), we find

$$\int_{\Gamma(t)} (u_{1R}(t) - u_{2R}(t))^+ + \liminf_{\epsilon \to 0} \int_0^t \int_{\mathcal{C}(s)} \nabla_{\bar{g}(s)}(v_{1R}(s) - v_{2R}(s)) \nabla_{\bar{g}(s)} E_R(s)\eta(s)$$
$$= \int_{\Gamma_0} (u_{1R}(0) - u_{2R}(0))^+. \tag{4.36}$$

[3]This holds because $uF_\epsilon(u)\nabla_\Gamma \cdot \mathbf{w} \to u\chi_{u \geq 0}\nabla_\Gamma \cdot \mathbf{w} = u^+\nabla_\Gamma \cdot \mathbf{w}$ and $uF_\epsilon(u)\nabla_\Gamma \cdot \mathbf{w} \leq C|u|$ is integrable, so the DCT applies.

The elliptic form in (4.36) becomes

$$\begin{aligned}
&\int_{\mathcal{C}(s)} \nabla_{\bar{g}(s)}(v_{1R}(s) - v_{2R}(s))\nabla_{\bar{g}(s)}E_R(s)\eta(s) \\
&= \frac{1}{\epsilon}\int_{\mathcal{C}(s)} \nabla_{\bar{g}(s)}(v_{1R}(s) - v_{2R}(s))\nabla_{\bar{g}(s)}T_\epsilon(\beta^{-1}(v_{1R}(s)) - \beta^{-1}(v_{2R}(s)))^+ \\
&= \frac{1}{\epsilon}\int_{B_\epsilon(s)} \nabla_{\bar{g}(s)}(v_{1R}(s) - v_{2R}(s))\nabla_{\bar{g}(s)}(\beta^{-1}(v_{1R}(s)) - \beta^{-1}(v_{2R}(s)))
\end{aligned}$$

where $B_\epsilon(s) := \{(x,y) \in \Gamma(s)\times[0,R] \mid 0 \le \beta^{-1}(v_{1R}(s,x,y)) - \beta^{-1}(v_{2R}(s,x,y)) < \epsilon\}$. Note that

$$\begin{aligned}
\nabla_{\bar{g}(s)}(\beta^{-1}(v_{1R}) - \beta^{-1}(v_{2R})) &= (\beta^{-1})'(v_{1R})\nabla_{\bar{g}(s)}v_{1R} - (\beta^{-1})'(v_{2R})\nabla_{\bar{g}(s)}v_{2R} \\
&= (\beta^{-1})'(v_{1R})(\nabla_{\bar{g}(s)}v_{1R} - \nabla_{\bar{g}(s)}v_{2R}) \\
&\quad + ((\beta^{-1})'(v_{1R}) - (\beta^{-1})'(v_{2R}))\nabla_{\bar{g}(s)}v_{2R},
\end{aligned}$$

so the above is

$$\begin{aligned}
&\int_{\mathcal{C}(s)} \nabla_{\bar{g}(s)}(v_{1R}(s) - v_{2R}(s))\nabla_{\bar{g}(s)}E_R(s)\eta(s) \\
&\ge \frac{1}{\epsilon}\int_{B_\epsilon(s)} ((\beta^{-1})'(v_{1R}(s)) - (\beta^{-1})'(v_{2R}(s)))\nabla_{\bar{g}(s)}(v_{1R}(s) - v_{2R}(s))\nabla_{\bar{g}(s)}v_{2R}.
\end{aligned} \tag{4.37}$$

The right hand side of this can be estimated as follows:

$$\begin{aligned}
&\left|\frac{1}{\epsilon}\int_{B_\epsilon(s)} ((\beta^{-1})'(v_{1R}(s)) - (\beta^{-1})'(v_{2R}(s)))\nabla_{\bar{g}(s)}(v_{1R}(s) - v_{2R}(s))\nabla_{\bar{g}(s)}v_{2R}(s)\right| \\
&\le \frac{1}{\epsilon}\int_{B_\epsilon(s)} |(\beta^{-1})'(v_{1R}(s)) - (\beta^{-1})'(v_{2R}(s))||\nabla_{\bar{g}(s)}(v_{1R}(s) - v_{2R}(s))||\nabla_{\bar{g}(s)}v_{2R}(s)| \\
&\le \frac{1}{\epsilon}\left\|(\beta^{-1})''\right\|_\infty \int_{B_\epsilon(s)} |v_{1R}(s) - v_{2R}(s)||\nabla_{\bar{g}(s)}(v_{1R}(s) - v_{2R}(s))||\nabla_{\bar{g}(s)}v_{2R}(s)| \\
&\le \frac{1}{\epsilon}\left\|(\beta^{-1})''\right\|_\infty \left\|\beta'\right\|_\infty \\
&\quad \times \int_{B_\epsilon(s)} |\beta^{-1}(v_{1R}(s)) - \beta^{-1}(v_{2R}(s))||\nabla_{\bar{g}(s)}(v_{1R}(s) - v_{2R}(s))||\nabla_{\bar{g}(s)}v_{2R}(s)| \\
&\le \left\|(\beta^{-1})''\right\|_\infty \left\|\beta'\right\|_\infty \int_{B_\epsilon(s)} |\nabla_{\bar{g}(s)}(v_{1R}(s) - v_{2R}(s))||\nabla_{\bar{g}(s)}v_{2R}(s)| \\
&= \left\|(\beta^{-1})''\right\|_\infty \left\|\beta'\right\|_\infty \int_{\mathcal{C}(s)} \chi_{B_\epsilon(s)}|\nabla_{\bar{g}(s)}(v_{1R}(s) - v_{2R}(s))||\nabla_{\bar{g}(s)}v_{2R}(s)|.
\end{aligned}$$

Now we must show that this expression tends to zero as ϵ tends to zero. Observe that

$$\chi_{B_\epsilon(s)}(z) \to \chi_{\{(x,y)\in\mathcal{C}_R(s)|\beta^{-1}(v_{1R}(s,x,y))-\beta^{-1}(v_{2R}(s,x,y))=0\}}(z) \tag{4.38}$$

pointwise a.e. $z \in C_R(s)$ and the integrand above is obviously bounded an integrable function, so the DCT applies and we obtain

$$\lim_{\epsilon\to 0}\int_{\mathcal{C}(s)} \chi_{B_\epsilon(s)}|\nabla_{\bar{g}(s)}(v_{1R}(s)-v_{2R}(s))||\nabla_{\bar{g}(s)}v_{2R}(s)| = \\ \int_{\mathcal{C}(s)} \chi_{\{z\in C_R(s)|\beta^{-1}(v_{1R}(s,z))-\beta^{-1}(v_{2R}(s,z))=0\}(z)}|\nabla_{\bar{g}(s)}(v_{1R}(s)-v_{2R}(s))||\nabla_{\bar{g}(s)}v_{2R}(s)|. \tag{4.39}$$

The set in the indicator function in the limit above is

$$\begin{aligned}&\{z \in C_R(s) \mid \beta^{-1}(v_{1R}(s,z)) - \beta^{-1}(v_{2R}(s,z)) = 0\}\\ &\quad = \{z \in C_R(s) \mid v_{1R}(s,z) - v_{2R}(s,z) = 0\},\end{aligned}$$

and $\nabla_{\bar{g}(s)}(v_{1R}(s)-v_{2R}(s))|_{\{v_{1R}(s)-v_{2R}(s)=0\}} = 0$ a.e. on $[0,R]\times\Gamma(s)$, by a theorem of Stampacchia [116] (see §4.3.3), so if $\{\beta^{-1}(v_{1R}(s))-\beta^{-1}(v_{2R}(s)) = 0\}$ has positive measure, then the integral on the right hand side of (4.39) vanishes. So then let us suppose that $\beta^{-1}(v_{1R})-\beta^{-1}(v_{2R}) = 0$ only on a set of measure zero. In this case, the right hand side of (4.38) is exactly 0, so again the right hand side vanishes. This implies in (4.37) that

$$\liminf_{\epsilon\to 0}\int_{\mathcal{C}(s)} \nabla_{\bar{g}(s)}(v_{1R}(s)-v_{2R}(s))\nabla_{\bar{g}(s)}E_R(s)\eta(s) \geq 0.$$

(This follows from the fact that if $a_\epsilon \geq b_\epsilon$ and $|b_\epsilon| \leq c_\epsilon$ and $c_\epsilon \to 0$, then $\liminf a_e \geq 0$[4]). Plugging this back into (4.36), we obtain for any t

$$\int_{\Gamma(t)} (u_{1R}(t)-u_{2R}(t))^+ \leq \int_{\Gamma_0} (u_{1R}(0)-u_{2R}(0))^+. \tag{4.40}$$

Now, by the work in the previous subsections, we have (amongst others) the uniform in R bound

$$\|u_{1R}\|_{L^2_{W^{1/2,2}}} + \|\dot{u}_{1R}\|_{L^2_{W^{-1/2,2}}} + \|u_{2R}\|_{L^2_{W^{1/2,2}}} + \|\dot{u}_{2R}\|_{L^2_{W^{-1/2,2}}} < C \quad \text{for all } R > 0$$

[4]We have $\liminf a_\epsilon \geq \liminf b_\epsilon \geq \liminf -|b_\epsilon| = -\limsup|b_\epsilon| \geq 0$ as $\limsup|b_\epsilon| \leq \limsup c_\epsilon = 0$.

and we have a subsequence $g(R)$ such that $u_{1g(R)} \to u_1$ in $L^2_{L^2}$, where u_1 is the solution of the non-degenerate problem with initial data u_0^1 (this is what we showed in the previous subsections). It follows that there is a subsequence $g(h(R))$ of $g(R)$ such that for almost all t, $u_{1g(h(R))}(t) \to u_1(t)$ in $L^2(\Gamma(t))$.

Since the uniform bounds above hold for all $R > 0$, $u_{2g(h(R))}$ is also bounded, and thus, by the same arguments, there is a further subsequence $u_{2g(h(l(R)))}(t) \to u_2(t)$ in $L^2(\Gamma(t))$ for almost all t. Note that also $u_{1g(h(l(R)))}(t) \to u_1(t)$ in $L^2(\Gamma(t))$ since subsequences share the same limit. Therefore, we can pass to the limit in (4.40) with R chosen to be the subsequence $g(h(l(R)))$ and we will obtain

$$\int_{\Gamma(t)} (u_1(t) - u_2(t))^+ \leq \int_{\Gamma_0} (u_0^1 - u_0^2)^+.$$

for almost all t. This concludes the proof of Theorem 4.1.6.

4.6 The fractional porous medium equation: proof of Theorem 4.1.4

We pick a sequence (see [51, p. 102]) of C^∞-approximations $\{\Psi_k\}_{k\in\mathbb{N}}$ of Ψ such that $\Psi_k(0) = 0$,

$$\begin{aligned}\frac{1}{C_k} \leq &\Psi_k' \leq k\\ &\Psi_k^{-1} \to \Psi^{-1} \qquad \text{in } C^0_{\text{loc}}(\mathbb{R})\\ &|\Psi_k^{-1}(r)| \leq C_1|r| + C_2\\ \frac{1}{k} \leq &(\Psi_k^{-1})' \leq C_k\\ &|(\Psi_k^{-1})''| \leq C_k.\end{aligned}$$

The previous section gives us a sequence $u_k \in \mathbb{W}(W^{1/2,2}, W^{-1/2,2})$ satisfying

$$\langle \dot{u}_k(t), \eta(t)\rangle + \int_{\Gamma(t)} u_k(t)\eta(t)\nabla_\Gamma \cdot \mathbf{w} + \int_0^\infty \int_{\Gamma(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))\nabla_{\bar{g}(t)}E(t)\eta(t) = 0. \tag{4.41}$$

4.6.1 Uniform estimates (in k)

Now we obtain appropriate estimates independent of k and pass to the limit for the last time. The idea is as follows:

- A weak maximum principle should yield

$$\|u_k\|_{L^\infty_{L^\infty}} + \|\Psi_k(u_k)\|_{L^\infty_{L^\infty}} < C.$$

- Testing with $\Psi_k(u_k)$ should give

$$\left\|\nabla_{\bar{g}}\overline{\mathbb{E}}(\Psi_k(u_k))\right\|_{L^2_{L^2(\mathcal{C})}} \leq C$$

  which will imply

$$\|\Psi_k(u_k)\|_{L^2_{W^{1/2,2}}} + \|\dot{u}_k\|_{L^2_{W^{-1/2,2}}} \leq C.$$

- These bounds imply $u_k \rightharpoonup u$ in $L^2_{L^2}$, $u_k \to u$ in $L^2_{W^{-1/2,2}}$, $\Psi_k(u_k) \rightharpoonup \Psi(u)$ in $L^2_{W^{1/2,2}}$ and $\nabla_{\bar{g}}\overline{\mathbb{E}}(\Psi_k(u_k)) \rightharpoonup \nabla_{\bar{g}}\overline{\mathbb{E}}(\Psi(u))$ in $L^2_{L^2(\mathcal{C})}$ (these identifications of the limits need to be proved, of course), which is enough to obtain existence.

We now justify this rigorously. For the maximum principle, in order to negate the effect of the lower order term involving $\nabla_\Gamma \cdot \mathbf{w}$, let us set $w_k(t) = u_k(t)e^{-\lambda t}$ and pick $\eta = (w_k - M)^+$ where $M := \|u_0\|_{L^\infty(\Gamma_0)}$. Note that $\dot{u}_k(t) = e^{\lambda t}(\dot{w}_k(t) + \lambda w_k(t))$. The equation (4.41) becomes

$$\begin{aligned}\langle \dot{w}_k(t), (w_k(t) - M)^+\rangle &+ \int_{\Gamma(t)} w_k(t)(w_k(t) - M)^+(\lambda + \nabla_\Gamma \cdot \mathbf{w}) \\ &+ e^{-\lambda t}\int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))\nabla_{\bar{g}(t)}E(t)(w_k(t) - M)^+ = 0.\end{aligned} \tag{4.42}$$

Now, we would like to pick the extension of $(w_k - M)^+$ to be

$$\left(\Psi_k^{-1}(\mathbb{E}(\Psi_k(u_k) - \overline{\Psi_k(u_k)}))e^{-\lambda t} - M\right)^+,$$

but this is not possible since $\Psi_k^{-1}(\mathbb{E}(\Psi_k(u_k) - \overline{\Psi_k(u_k)}))e^{-\lambda t} - M$ is not square integrable. Therefore, defining

$$g(u_k, \rho) := \mathbb{E}(\Psi_k(u_k) - \overline{\Psi_k(u_k)}) + \psi_\rho \overline{\Psi_k(u_k)},$$

we pick $E(w_k - M)^+ = \left(\Psi_k^{-1}(g(u_k,\rho))e^{-\lambda t} - M\psi_\rho\right)^+ \in L^2_{H^1(\mathcal{C})}$ which satisfies $\mathbb{T}E\eta = (w_k(t) - M)^+$ and

$$\nabla_{\bar{g}(t)} E\eta = \begin{cases} (\Psi_k^{-1})'(g(u_k,\rho))e^{-\lambda t}\nabla_{\bar{g}(t)} g(u_k,\rho) - M\psi_\rho' & : \Psi_k^{-1}(g(u_k,\rho))e^{-\lambda t} \geq M\psi_\rho \\ 0 & : \text{otherwise} \end{cases}$$

so that the gradient term in (4.42) on $\{\Psi_k^{-1}(g(u_k,\rho))e^{-\lambda t} - M\psi_\rho \geq 0\}$ is

$$\begin{aligned}
&\int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))\nabla_{\bar{g}(t)}E(t)\eta(t) \\
&= \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))\Big((\Psi_k^{-1})'(g(u_k,\rho))e^{-\lambda t}(\nabla_{\bar{g}(t)}\mathbb{E}(\Psi_k(u_k(t)) - \overline{\Psi_k(u_k(t))}) \\
&\qquad\qquad + \psi_\rho'\overline{\Psi_k(u_k(t))}) - M\psi_\rho'\Big) \\
&= \int_{\mathcal{C}(t)} (\Psi_k^{-1})'(g(u_k,\rho))e^{-\lambda t}\left[|\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))|^2 + \partial_y\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))\psi_\rho'\overline{\Psi_k(u_k(t))}\right] \\
&\quad - M\int_{\mathcal{C}(t)} \partial_y\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))\psi_\rho' \\
&\geq e^{-\lambda t}k^{-1}\int_{\mathcal{C}(t)} |\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))|^2 - e^{-\lambda t}C_k\int_{\mathcal{C}(t)} |\partial_y\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))||\psi_\rho'\overline{\Psi_k(u_k(t))}|,
\end{aligned} \tag{4.43}$$

and the last term vanished in the final inequality because for a.a. t,

$$\int_{\Gamma(t)} \partial_y\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))\psi_\rho' = \psi_\rho'\int_{\Gamma(t)} \partial_y\mathcal{E}_t(\Psi_k(u_k(t)) - \overline{\Psi_k(u_k(t))}) = 0$$

since $\mathcal{E}_t(\Psi_k(u_k(t)) - \overline{\Psi_k(u_k(t))})$ has mean value zero. Now, we have that the last integral on the right hand side of (4.43) is

$$\begin{aligned}
&\int_{\mathcal{C}(t)} |\partial_y\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))||\psi_\rho'\overline{\Psi_k(u_k(t))}| \\
&\leq \int_{\mathcal{C}(t)} \epsilon|\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))|^2 + C_\epsilon|\psi_\rho'\overline{\Psi_k(u_k(t))}|^2 \\
&\leq \int_{\mathcal{C}(t)} \epsilon|\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))|^2 + C_\epsilon C|\Gamma(t)||\overline{\Psi_k(u_k(t))}|^2\int_\rho^{2\rho}\frac{1}{\rho^2} \\
&= \int_{\mathcal{C}(t)} \epsilon|\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))|^2 + C_\epsilon C|\Gamma(t)||\overline{\Psi_k(u_k(t))}|^2\frac{1}{\rho},
\end{aligned}$$

hence if ϵ is small enough, (4.43) implies

$$\begin{aligned}
&e^{-\lambda t}\int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))\nabla_{\bar{g}(t)}E(t)\eta(t)\\
&\geq \frac{1}{2}e^{-2\lambda t}C_1(k)\int_{\mathcal{C}(t)} |\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))|^2 - \frac{1}{2}e^{-2\lambda t}C_2(k)|\Gamma(t)||\overline{\Psi_k(u_k(t))}|^2\frac{1}{\rho}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\langle \dot{w}_k(t), (w_k(t)-M)^+\rangle + \int_{\Gamma(t)} w_k(t)(w_k(t)-M)^+(\lambda + \nabla_\Gamma\cdot\mathbf{w})\\
&+\chi_{\{\Psi_k^{-1}(g(u_k,\rho))e^{-\lambda t}-M\psi_\rho\geq 0\}}\frac{1}{2}e^{-2\lambda t}\\
&\times\left(C_1(k)\int_0^\infty\int_{\Gamma(t)} |\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi_k(u_k(t)))|^2 - C_2(k)|\Gamma(t)||\overline{\Psi_k(u_k(t))}|^2\frac{1}{\rho}\right)\leq 0.
\end{aligned}$$

Choosing $\lambda := \|\nabla_\Gamma\cdot\mathbf{w}\|_\infty$ and sending $\rho\to\infty$, observing that

$$\chi_{\{\Psi_k^{-1}(g(u_k,\rho))e^{-\lambda t}-M\psi_\rho\geq 0\}}\frac{1}{2}e^{-2\lambda t}C_2(k)|\Gamma(t)||\overline{\Psi_k(u_k(t))}|^2\frac{1}{\rho}\to 0$$

(since of course the characteristic function is bounded above by one), we get

$$\begin{aligned}
\frac{d}{dt}\int_{\Gamma(t)}((w_k(t)-M)^+)^2 - \int_{\Gamma(t)}((w_k(t)-M)^+)^2\nabla_\Gamma\cdot\mathbf{w} &= 2\langle \dot{w}_k(t), (w_k(t)-M)^+\rangle\\
&\leq 0.
\end{aligned}$$

Gronwall's inequality implies boundedness of w_k and hence

$$\operatorname*{ess\,sup}_{t\in[0,T]}\|u_k(t)\|_{L^\infty(\Gamma(t))} + \operatorname*{ess\,sup}_{t\in[0,T]}\|\Psi_k(u_k(t))\|_{L^\infty(\Gamma(t))}\leq C \tag{4.44}$$

independent of k. The second L^∞ bound holds because

$$\operatorname*{ess\,sup}_{t\in[0,T]}\|\Psi_k(u_k(t))\|_{L^\infty(\Gamma(t))}\leq \max\left(|\Psi_k(C)|, |\Psi_k(-C)|\right)$$

since Ψ_k is increasing, and the right hand side is bounded (in $\mathbb{R}$) because

$$\max\left(|\Psi_k(C)|, |\Psi_k(-C)|\right)\to\max\left(|\Psi(C)|, |\Psi(-C)|\right),$$

and every convergent sequence in $\mathbb{R}$ is bounded.

Now we focus on obtaining a bound on $\|\Psi_k(u_k)\|_{L^2_{W^{1/2,2}}}$.

Definition 4.6.1. Define

$$H_k(r) = \int_0^r \Psi_k(s)\, \mathrm{d}s \qquad \text{and} \qquad G_k(r) = \int_0^r \Psi_k^{-1}(s)\, \mathrm{d}s$$

and

$$H(r) = \int_0^r \Psi(s)\, \mathrm{d}s \qquad \text{and} \qquad G(r) = \int_0^r \Psi^{-1}(s)\, \mathrm{d}s.$$

If $u \in L^2(M)$, then $G_k(u) \in L^1(M)$ and $H_k(\Psi_k^{-1}(u)) \in L^1(M)$; to see this, note that G_k is continuous, thus $x \mapsto G_k(u(x))$ is measurable, and

$$|G_k(u)| \leq \max\{|\Psi_k^{-1}(u)||u|, |\Psi_k^{-1}(-u)||u|) \leq (C_1|u| + C_2)|u| \tag{4.45}$$

which is obviously integrable. Since

$$H_k(\Psi_k^{-1}(u)) = u\Psi_k^{-1}(u) - G_k(u) \leq (C_3|u| + C_4)|u|, \tag{4.46}$$

we also have that $H_k(\Psi_k^{-1}(u))$ is in $L^1(M)$.

Remark 4.6.2. We could have generalised the porous medium nonlinearity $\Psi(r) = |r|^{m-1}r$ to simply having Ψ as a continuous increasing function. In this case Ψ is no longer invertible so we would have to use Legendre transforms [51].

Test the equation with $\eta = \Psi_k(u_k)$ and use the integration by parts formula of Lemma 4.3.11:

$$\begin{aligned}&\int_{\Gamma(T)} H_k(u_k(T)) - \int_{\Gamma_0} H_k(u_0) - \int_0^T \int_{\Gamma(t)} H_k(u_k)\nabla_\Gamma \cdot \mathbf{w} \\ &+ \int_0^T \int_{\Gamma(t)} u_k \Psi_k(u_k) \nabla_\Gamma \cdot \mathbf{w} + \int_0^T \int_{\mathcal{C}(t)} \nabla_{\bar{g}(t)} \overline{\mathcal{E}}_t(\Psi_k(u_k)) \nabla_{\bar{g}(t)} E(t)\Psi_k(u_k(t)) = 0.\end{aligned}$$

Since H_k is always positive (it is the integral of function which is positive on $(0, \infty)$ and negative on $(-\infty, 0)$), the first term can be neglected, and if we pick $E(\Psi_k(u_k)) = \mathbb{E}(\Psi_k(u_k) - \overline{\Psi_k(u_k)}) + \psi_\rho \overline{\Psi_k(u_k)}$, we obtain

$$\begin{aligned}&\int_0^T \int_{\mathcal{C}(t)} |\nabla_{\bar{g}(t)} \mathcal{E}_t(\Psi_k(u_k) - \overline{\Psi_k(u_k)})|^2 + \partial_y \mathcal{E}_t(\Psi_k(u_k) - \overline{\Psi_k(u_k)}) \psi_\rho' \overline{\Psi_k(u_k)} \\ &\leq \int_{\Gamma_0} H_k(u_0) + \int_0^T \int_{\Gamma(t)} H_k(u_k) \nabla_\Gamma \cdot \mathbf{w} - \int_0^T \int_{\Gamma(t)} u_k \Psi_k(u_k) \nabla_\Gamma \cdot \mathbf{w}.\end{aligned}$$

The second term on the LHS disappears since the harmonic extension of a mean value zero function has mean value zero too. Then we finally get after using (4.46) that $|H_k(u_k)| \leq C_1 \|\Psi_k(u_k)\|^2_{L^\infty_{L^\infty}} + C_2$. This takes care of the second term on the right hand side, and as for the initial data, we note that

$$\left|\int_{\Gamma_0} H_k(u_0)\right| \leq C_1 \int_{\Gamma_0} |\Psi_k(u_0)|^2 + C_2|\Gamma_0| \leq C_1|\Gamma_0| \|\Psi_k(u_0)\|^2_{L^\infty(\Gamma_0)} + C_2|\Gamma_0|$$

and $\|\Psi_k(u_0)\|_{L^\infty(\Gamma_0)} \leq \max(|\Psi_k(\|u_0\|_{L^\infty(\Gamma_0)})|, -|\Psi_k(\|u_0\|_{L^\infty(\Gamma_0)})|)$, and the right hand side is bounded, just like before. Thus

$$\left\|\nabla_{\bar{g}}\mathbb{E}(\Psi_k(u_k) - \overline{\Psi_k(u_k)})\right\|_{L^2_{L^2(\mathcal{C})}} \leq C.$$

We also have, using $\Psi_k(u_k) = \overline{\mathbb{T}}\mathbb{E}(\Psi_k(u_k))$,

$$\begin{aligned}
\|\Psi_k(u_k)\|_{L^2_{W^{1/2,2}}} &\leq C_1 \left\|\overline{\mathbb{E}}(\Psi_k(u_k))\right\|_{L^2_X} && \text{(by Lemma 4.3.9)} \\
&= C_1 \left(\left\|\nabla_{\bar{g}}\mathbb{E}(\Psi_k(u_k) - \overline{\Psi_k(u_k)})\right\|_{L^2_{L^2(\mathcal{C})}} + \|\Psi_k(u_k)\|_{L^2_{L^2}} \right) \\
&\leq C_2.
\end{aligned}$$

Finally, integrating and rearranging (4.41):

$$\int_0^T \langle \dot{u}_k(t), \eta(t)\rangle \leq \|\nabla_\Gamma \cdot \mathbf{w}\| \|u_k\|_{L^2_{L^2}} \|\eta\|_{L^2_{L^2}} + \left\|\nabla_{\bar{g}}\overline{\mathbb{E}}(\Psi_k(u_k))\right\|_{L^2_{L^2(\mathcal{C})}} \|\nabla_{\bar{g}}E\eta\|_{L^2_{L^2(\mathcal{C})}}$$

and choosing $E\eta = \mathscr{Z}_\rho \overline{\mathbb{E}}_\rho \eta$ for some $\rho > 1$ and using

$$\|\nabla_{\bar{g}}E\eta\|_{L^2_{L^2(\mathcal{C})}} = \left\|\nabla_{\bar{g}}\mathscr{Z}_\rho\overline{\mathbb{E}}_\rho\eta\right\|_{L^2_{L^2(\mathcal{C})}} = \left\|\nabla_{\bar{g}}\overline{\mathbb{E}}_\rho\eta\right\|_{L^2_{L^2(\mathcal{C}_\rho)}} \leq C \|\eta\|_{L^2_{W^{1/2,2}}}$$

with the last inequality by Lemma 4.4.8, it easily follows that

$$\|\dot{u}_k\|_{L^2_{W^{-1/2,2}}} \leq C \tag{4.47}$$

independent of k. Therefore, we have

$$\begin{aligned}
u_k &\rightharpoonup u && \text{in } L^2_{L^2} \\
u_k &\to u && \text{in } L^2_{W^{-1/2,2}} \\
v_k := \Psi_k(u_k) &\rightharpoonup v && \text{in } L^2_{W^{1/2,2}} \\
\nabla_{\bar{g}}\mathbb{E}(v_k - \bar{v}_k) &\rightharpoonup \alpha && \text{in } L^2_{L^2(C)}
\end{aligned} \tag{4.48}$$

with the strong convergence by Aubin–Lions. Now the question is whether $v = \Psi(u)$. If it were so, then we can also identify α: indeed, we know that the map $\mathbb{G}\colon L^2_{W^{1/2,2}} \to L^2_{L^2(\mathcal{C})}$ defined by $\mathbb{G}w = \nabla_{\bar{g}}\mathbb{E}(w - \bar{w})$ is linear and also continuous by Lemma 4.4.5:

$$\|\mathbb{G}w\|_{L^2_{L^2(\mathcal{C})}} \leq C_1 \|w - \bar{w}\|_{L^2_{W^{1/2,2}}} \leq C_2 \|w\|_{L^2_{W^{1/2,2}}},$$

and this implies that $\mathbb{G}v_k \rightharpoonup \mathbb{G}\Psi(u)$ in $L^2_{L^2(\mathcal{C})}$, i.e.,

$$\nabla_{\bar{g}}\mathbb{E}(v_k - \bar{v}_k) \rightharpoonup \nabla_{\bar{g}}\mathbb{E}(\Psi(u) - \overline{\Psi(u)}) \quad \text{in } L^2_{L^2(\mathcal{C})}.$$

Now we show that indeed $v = \Psi(u)$.

4.6.2 Identification of $v \equiv \Psi(u)$

We shall use the theory of subdifferentials, see §4.A.1 for a reminder. Let us define

$$J_k(v) = \begin{cases} \int_0^T \int_{\Gamma(t)} G_k(v) & \text{if } G_k(v) \in L^1_{L^1} \\ 0 & \text{otherwise,} \end{cases}$$
$$J(v) = \begin{cases} \int_0^T \int_{\Gamma(t)} G(v) & \text{if } G(v) \in L^1_{L^1} \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $v \in L^2_{L^2}$ then $G_k(v)$, $G(v) \in L^1_{L^1}$ (see (4.45)).

Lemma 4.6.3. The map

$$v \mapsto \int_0^T \int_{\Gamma(t)} G(v)$$

from $L^2_{L^2}$ into $\mathbb{R}$ is lower semicontinuous.

Proof. First, observe that $G\colon \mathbb{R} \to \mathbb{R}$ is convex, proper and continuous, hence (for example by adapting Proposition 8.1 in [112, Chapter II]) the map (which is well-defined, for example, see (4.45))

$$w \mapsto \int_{\Gamma(t)} G(w) \quad \text{for } w \in L^2(\Gamma(t))$$

is lower semicontinuous for each fixed t. If $v_n \to v$ in $L^2_{L^2}$, we have $v_{n_j}(t) \to v(t)$ in $L^2(\Gamma(t))$ for almost all t, so

$$\int_{\Gamma(t)} G(v(t)) \leq \liminf_{n_j \to \infty} \int_{\Gamma(t)} G(v_{n_j}(t)). \tag{4.49}$$

Integrating (4.49), and since $\int_{\Gamma(t)} G(v_{n_j}(t)) \geq 0$ and the map $t \mapsto \int_{\Gamma(t)} G(v_{n_j}(t)) = \int_{\Gamma_0} G(\tilde{v}_{n_j}(t)) J_t^0$ is measurable, we can apply Fatou's lemma to give

$$\int_0^T \int_{\Gamma(t)} G(v(t)) \leq \int_0^T \liminf_{n_j \to \infty} \int_{\Gamma(t)} G(v_{n_j}(t)) \leq \liminf_{n_j \to \infty} \int_0^T \int_{\Gamma(t)} G(v_{n_j}(t)).$$

Thus far we have shown that for any sequence $v_n \to v$ converging in $L^2_{L^2}$, $J(v) \leq \liminf_{j \to \infty} J(v_{n_j})$ holds for a subsequence n_j. Now, if $v_n \to v$ in $L^2_{L^2}$, then it follows that there is a subsequence v_{n_j} such that

$$\liminf_{n \to \infty} J(v_n) = \lim_{j \to \infty} J(v_{n_j}) \tag{4.50}$$

by definition of the lim inf (J is non-negative, so either $\liminf J(v_n) = \infty$ or $\liminf J(v_n) = C \geq 0$; the former case makes the problem trivial so we can discount it). We know that there is a subsequence n_{j_k} of n_j such that $J(v) \leq \liminf_{k \to \infty} J(v_{n_{j_k}}) = \lim_{j \to \infty} J(v_{n_j}) = \liminf_{n \to \infty} J(v_n)$ with the first equality because the limit of $J(v_{n_{j_k}})$ is the same as the limit of $J(v_{n_j})$ and the second equality from (4.50). $\square$

Lemma 4.6.4. We have $u = \Psi^{-1}(v)$.

Proof. It follows that J_k and J are convex (since G_k and G are convex) and positive (see [14, §2.4]). Lemma 4.A.9 tells us that if the Gâteaux derivative exists at a particular point, then the set of subdifferentials coincide with the set of Gâteaux derivatives at that particular point. Indeed, the subdifferentials are

$$\begin{aligned} \partial J_k(v_k) &= \{w \in L^2_{L^2} \mid w = \Psi_k^{-1}(v_k) \text{ in } L^2_{L^2}\} \\ \partial J(v) &= \{w \in L^2_{L^2} \mid w = \Psi^{-1}(v) \text{ in } L^2_{L^2}\}. \end{aligned}$$

To see this, for example, taking v, $h \in L^2_{L^2}$,

$$\begin{aligned} \frac{J(v + \lambda h) - J(v)}{\lambda} &= \int_0^T \int_{\Gamma(t)} \frac{G(v + \lambda h) - G(v)}{\lambda} \\ &= \frac{1}{\lambda} \int_0^T \int_{\Gamma(t)} \int_0^1 \frac{d}{ds}(G(v + \lambda h s)) \,\mathrm{d}s \\ &= \int_0^T \int_{\Gamma(t)} \int_0^1 \Psi^{-1}(v + \lambda h s) h \,\mathrm{d}s, \end{aligned}$$

and now we can take limits (using DCT):

$$\lim_{\lambda\to 0}\frac{J(v+\lambda h)-J(v)}{\lambda}=\int_0^T\int_{\Gamma(t)}\int_0^1\Psi^{-1}(v)h\;\mathrm{d}s=(\Psi^{-1}(v),h)_{L^2_{L^2}}.$$

So the Gâteaux derivative of $J(v)$ is $\Psi^{-1}(v)$. By Definition 4.A.8, since $\Psi_k^{-1}(v_k)\in\partial J_k(v_k)$, for all $w\in L^2_{L^2}$,

$$\int_0^T\int_{\Gamma(t)}G_k(v_k)+\int_0^T\int_{\Gamma(t)}\Psi_k^{-1}(v_k)w\leq\int_0^T\int_{\Gamma(t)}G_k(w)+\int_0^T\int_{\Gamma(t)}\Psi_k^{-1}(v_k)v_k. \tag{4.51}$$

We want to pass to the limit in this equation using (4.48). Let us consider each term in turn.

1. For the first term on the right hand side: we have for almost all t and almost all $x\in\Gamma(t)$,

   $$G_k(w(t,x))=\int_0^{w(t,x)}\Psi_k^{-1}(s)\to\int_0^{w(t,x)}\Psi^{-1}(s)=G(w(t,x))$$

   by the convergence of $\Psi_k^{-1}\to\Psi^{-1}$. We also have by (4.45) that

   $$|G_k(w(t,x))|\leq C(|w(t,x)|^2+|w(t,x)|).$$

   The right hand side is in $L^1_{L^1}$, so by the DCT, $G_k(w)\to G(w)$ in $L^1_{L^1}$, which obviously implies that

   $$\int_0^T\int_{\Gamma(t)}G_k(w)\to\int_0^T\int_{\Gamma(t)}G(w).$$

2. For the second term on the right hand side, since $u\in L^2_{L^2}$,

   $$(u_k,v_k)_{L^2_{L^2}}=\langle\Psi_k^{-1}(v_k),v_k\rangle_{L^2_{W^{-1/2,2}},L^2_{W^{1/2,2}}}\to\langle u,v\rangle_{L^2_{W^{-1/2,2}},L^2_{W^{1/2,2}}}=(u,v)_{L^2_{L^2}}.$$

3. For the first term on the left hand side, we first show an intermediary step, that

   $$\lim_{m\to\infty}\int_0^T\int_{\Gamma(t)}G_k(v_k)-G(v_k)=0. \tag{4.52}$$

- To see this, note that for a.a. t and a.e. x,

$$\begin{aligned}|G_k(v_k(t,x)) - G(v_k(t,x))| &= \left| \int_0^{v_k(t,x)} (\Psi_k^{-1}(s) - \Psi^{-1}(s)) \right| \\ &\leq \sup_{s\in[-\|v_k\|_\infty, \|v_k\|_\infty]} \|v_k\|_\infty |\Psi_k^{-1}(s) - \Psi^{-1}(s)| \\ &\leq C \sup_{s\in[-C,C]} |\Psi_k^{-1}(s) - \Psi^{-1}(s)|,\end{aligned}$$

  hence

$$\begin{aligned}\left| \int_0^T \int_{\Gamma(t)} G_k(v_k) - G(v_k) \right| &\leq \int_0^T \int_{\Gamma(t)} |G_k(v_k) - G(v_k)| \\ &\leq |\Gamma| T \sup_{s\in[-C,C]} |\Psi_k^{-1}(s) - \Psi^{-1}(s)| C \to 0\end{aligned}$$

  by the convergence of $\Psi_k^{-1} \to \Psi^{-1}$.

By weak lower semicontinuity of the map $v \mapsto \int_0^T \int_{\Gamma(t)} G(v)$ (Lemma 4.6.3), we have

$$\int_0^T \int_{\Gamma(t)} G(v) \leq \liminf_{k\to\infty} \int_0^T \int_{\Gamma(t)} G(v_k) = \liminf_{k\to\infty} \int_0^T \int_{\Gamma(t)} G_k(v_k)$$

with the equality by (4.52).

4. The second term on the left hand side is obvious.

Using the above facts, we can pass to the limit in (4.51) to get

$$\int_0^T \int_{\Gamma(t)} G(v) + \int_0^T \int_{\Gamma(t)} uw \leq \int_0^T \int_{\Gamma(t)} G(w) + \int_0^T \int_{\Gamma(t)} uv,$$

which is exactly the statement $u \in \partial J(v)$, i.e., $u = \Psi^{-1}(v)$. □

That $u \in L^\infty_{L^\infty}$ follows like in §3.3.2.

Conclusion Integrating (4.41) by parts over time and letting $\eta \in \mathbb{W}(W^{1/2,2}, L^2)$ with $\eta(T) = 0$, the equation we want to pass to the limit to in is

$$-\int_0^T \int_{\Gamma(t)} \dot{\eta}(t) u_k(t) + \int_0^T \int_0^\infty \int_{\Gamma(t)} \nabla_{\bar{g}(t)} \overline{\mathcal{E}}_t(\Psi_k(u_k(t))) \nabla_{\bar{g}(t)} E(t)\eta(t) = \int_{\Gamma_0} u_0 \eta(0),$$

and doing so yields

$$-\int_0^T\int_{\Gamma(t)}\dot{\eta}(t)u(t)+\int_0^T\int_0^\infty\int_{\Gamma(t)}\nabla_{\bar{g}(t)}\overline{\mathcal{E}}_t(\Psi(u))\nabla_{\bar{g}(t)}E(t)\eta(t)=\int_{\Gamma_0}u_0\eta(0).$$

4.6.3 Contraction principle

We know that the solutions u_{1k} and u_{2k} of the non-degenerate problem (with non-linearity Ψ_k) and initial data u_0^1 and u_0^2 respectively satisfy

$$\int_{\Gamma(t)}(u_{1k}(t)-u_{2k}(t))^+\leq\int_{\Gamma_0}(u_0^1-u_0^2)^+\qquad\text{for all } k \tag{4.53}$$

by the contraction principle in Theorem 4.1.6. We have shown that there is a subsequence $f(k)$ such that $u_{1f(k)}$ converges to u_1, the solution of the fractional porous medium equation. The sequence $u_{2f(k)}$ is also bounded and so there exists a subsequence $u_{2f(\tilde{f}(k))}$ converging to u_2, and $u_{1f(\tilde{f}(k))}$ converges to u_1 too.

Now, for $i=1,2$, from (4.44) and (4.47),

$$\|\tilde{u}_{ik}\|_{L^\infty(0,T;L^\infty(\Gamma_0))}+\left\|\tilde{u}_{ik}'\right\|_{L^2(0,T;W^{-1/2,2}(\Gamma_0))}\leq C.$$

The sequences $u_{if(\tilde{f}(k))}$ also satisfy these bounds and therefore by Aubin–Lions (Theorem II.5.16 in [24]), for a subsequence of $f(\tilde{f}(k))$, which we will write as $g(k)$, we have $\tilde{u}_{1g(k)}\to\tilde{u}_1$ in $C^0([0,T];W^{-1/2,2}(\Gamma_0))$[5]. The sequence $\tilde{u}_{2g(k)}$ and its derivative is also bounded, so again there is a subsequence $\tilde{u}_{2g(h(k))}\to\tilde{u}_2$ in $C^0([0,T];W^{-1/2,2}(\Gamma_0))$. Therefore, we have

$$\begin{aligned}\tilde{u}_{1g(h(k))}(t)&\to\tilde{u}_1(t)\quad\text{in } W^{-1/2,2}(\Gamma_0)\\ \tilde{u}_{2g(h(k))}(t)&\to\tilde{u}_2(t)\quad\text{in } W^{-1/2,2}(\Gamma_0).\end{aligned}\tag{4.54}$$

By the uniform bound, we have for almost all t that $\left\|\tilde{u}_{1g\circ h(k)}(t)\right\|_{L^\infty(\Gamma_0)}\leq C$, which

[5]In fact, we first have $\tilde{u}_{1g(k)}\to w$ in $C^0([0,T];W^{-1/2,2}(\Gamma_0))$ for some w which we need to identify. Since $C^0([0,T];W^{-1/2,2}(\Gamma_0))\hookrightarrow L^2(0,T;W^{-1/2,2}(\Gamma_0))$, we have the weak convergence

$$\int_0^T\langle\tilde{u}_{1g(k)}(t),\tilde{v}(t)\rangle_0\to\int_0^T\langle w(t),\tilde{v}(t)\rangle_0$$

for all $\tilde{v}\in L^2(0,T;W^{1/2,2}(\Gamma_0))$. But we know $\tilde{u}_{1g(k)}\rightharpoonup\tilde{u}_1$ in $L^p(0,T;L^p(\Gamma_0))$ for any p (we may have to start off by passing to a subsequence first but it does not matter; we just call the subsequence $g(k)$), so the left hand side converges like

$$\int_0^T\langle\tilde{u}_{1g(k)}(t),\tilde{v}(t)\rangle_0=\int_0^T\int_{\Gamma_0}\tilde{u}_{ik}(t)\tilde{v}(t)\to\int_0^T\int_{\Gamma_0}\tilde{u}_1(t)\tilde{v}(t)=\int_0^T\langle\tilde{u}_1(t),\tilde{v}(t)\rangle_0$$

and so $\tilde{w}=\tilde{u}_1$.

gives for almost all t, $\tilde{u}_{1g\circ h\circ l_t(k)}(t) \overset{*}{\rightharpoonup} \theta_1(t)$ in $L^\infty(\Gamma_0)$. Again by the uniform bound, we have for almost all t $\left\|\tilde{u}_{2g\circ h\circ l_t(k)}(t)\right\|_{L^\infty(\Gamma_0)} \leq C$ for the subsequence. This yields $\tilde{u}_{2g\circ h\circ l_t\circ s_t(k)}(t) \overset{*}{\rightharpoonup} \theta_2(t)$ in $L^\infty(\Gamma_0)$ for a further subsequence. Note that we also have $\tilde{u}_{1g\circ h\circ l_t\circ s_t(k)}(t) \overset{*}{\rightharpoonup} \theta_1(t)$ in $L^\infty(\Gamma_0)$ since every subsequence has the same weak-star limit. We can identify $\theta_i(t) = \tilde{u}_i(t)$ thanks to the strong convergence (4.54). Thus,

$$\tilde{u}_{1g\circ h\circ l_t\circ s_t(k)}(t) - \tilde{u}_{2g\circ h\circ l_t\circ s_t(k)}(t) \rightharpoonup \tilde{u}_1(t) - \tilde{u}_2(t) \qquad \text{in } L^1(\Gamma_0).$$

Since $(\cdot)^+$ is a convex function, $I_t \colon L^1(\Gamma(t)) \to \mathbb{R}$ defined by $I_t(u) = \int_{\Gamma(t)} u^+$ is convex, and clearly it is also continuous. By a corollary of Mazur's lemma, I_t is weakly lower semicontinuous, which gives

$$\int_{\Gamma(t)} (u_1(t) - u_2(t))^+ \leq \int_{\Gamma_0} (u_0^1 - u_0^2)^+ \qquad \text{for almost all } t \in [0,T] \tag{4.55}$$

from (4.53). Now let us show that this holds for every $t \in [0,T]$. We know from above that for almost all $s \in [0,T]$,

$$\left\|\tilde{u}_{1g\circ h(k)}(s)\right\|_{L^\infty(\Gamma_0)} + \left\|\tilde{u}_{2g\circ h(k)}(s)\right\|_{L^\infty(\Gamma_0)} \leq C$$

with C independent of k and s. Let $t \in [0,T]$ be arbitrary and take a sequence $t_j \to t$ where each t_j is such that

$$\left\|\tilde{u}_{1g\circ h(k)}(t_j)\right\|_{L^\infty(\Gamma_0)} + \left\|\tilde{u}_{2g\circ h(k)}(t_j)\right\|_{L^\infty(\Gamma_0)} \leq C. \tag{4.56}$$

Since C is independent of j, we have $\tilde{u}_{1g\circ h(k)}(t_{b_k(j)}) \overset{*}{\rightharpoonup} a_{1k}$ in $L^\infty(\Gamma_0)$ as $j \to \infty$, and since for this subsequence, $\left\|\tilde{u}_{2g\circ h(k)}(t_{b_k(j)})\right\|_{L^\infty(\Gamma_0)} \leq C$, we have $\tilde{u}_{2g\circ h(k)}(t_{b_k\circ c_k(j)}) \overset{*}{\rightharpoonup} a_{2k}$. Note also that $\tilde{u}_{1g\circ h(k)}(t_{b_k\circ c_k(j)}) \overset{*}{\rightharpoonup} a_{1k}$. Since $\tilde{u}_{ig\circ h(k)} \in C^0([0,T]; W^{-1/2,2}(\Gamma_0))$, we have $\tilde{u}_{ig\circ h(k)}(t_{b_k\circ c_k(j)}) \to \tilde{u}_{ig\circ h(k)}(t)$ in $W^{-1/2,2}(\Gamma_0)$ for every t which allows us to identify a_{ik}, and we have

$$\begin{aligned} \tilde{u}_{1g\circ h(k)}(t_{b_k\circ c_k(j)}) &\overset{*}{\rightharpoonup} \tilde{u}_{1g\circ h(k)}(t) \\ \tilde{u}_{2g\circ h(k)}(t_{b_k\circ c_k(j)}) &\overset{*}{\rightharpoonup} \tilde{u}_{2g\circ h(k)}(t). \end{aligned} \tag{4.57}$$

The weak-star lower semi-continuity of norms with (4.56) and (4.57) give

$$\left\|\tilde{u}_{1g\circ h(k)}(t)\right\|_{L^\infty(\Gamma_0)} + \left\|\tilde{u}_{2g\circ h(k)}(t)\right\|_{L^\infty(\Gamma_0)} \leq C \qquad \text{for every } t \in [0,T].$$

Then the argument we used to prove (4.55) (which holds almost every t) can be repeated, and we will obtain the result for every t. Note that we obtain $u(t) \in$

$L^\infty(\Gamma(t))$ for all t as a side product of the above argument.

As for the conservation of mass, since $\int_{\Gamma(t)} u_k(t) = \int_{\Gamma_0} u_0$ holds for all t and all k, and since (for a subsequence) $u_k(t) \overset{*}{\rightharpoonup} u(t)$ for all t, we can easily pass to the limit.

4.7 Concluding remarks

The (non-fractional) porous medium equation on an evolving surface can be also tackled in this way, as a limit of approximations; of course the problem is easier in that case as we would not need §4.2, §4.4 and parts of §4.3, and the non-degenerate problem in §4.5 can be handled with a fixed point argument using the linear theory in §1, as done in 2 for a Stefan problem. We name a few of the many interesting open issues left. We required bounded initial data for the results above but the L^1-continuous dependence result leaves us in good position to extend the results to integrable data if we manage to obtain a smoothing effect (for which the work [18] by Bonforte and Grillo may be useful). There is also the fast diffusion or the singular case where $m \in (0, 1)$ which we have not addressed. A fundamental property enjoyed by solutions of the fractional porous medium equation on a stationary domain is regularity in time [44, Theorem 2.3], that is, the solution has a time derivative in L^1. In the stationary case, this regularity is obtained partially by a rescaling argument of [10] and using the L^1-continuous dependence applied to a solution and its rescaled version. This does not work in our setting since rescaled solutions live on a different evolving hypersurface, so the continuous dependence inequality cannot be applied. This result would be useful because it would allow us to study qualitative properties such as the effect the geometry of the hypersurface has on the solution. An obvious further extension is to study this theory of weak solutions with a general exponent in the fractional Laplacian $(-\Delta_{\Gamma(t)})^s$: for this of course [29] is the obvious starting point and the methodology we used in this chapter should work.

4.A Appendix

Proof of Lemma 4.5.3. For the boundedness, we have

$$\begin{aligned}
\|\beta(u)\|^2_{W^{1/2,2}(\Gamma)} &= \int_\Gamma |\beta(u)|^2 + \int_\Gamma \int_\Gamma \frac{|\beta(u(x)) - \beta(u(y))|^2}{|x-y|^n} \\
&\le \mathrm{Lip}(\beta)^2 \int_\Gamma |u|^2 + \mathrm{Lip}(\beta)^2 \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^2}{|x-y|^n} \\
&= \mathrm{Lip}(\beta)^2 \|u\|^2_{W^{1/2,2}(\Gamma)} .
\end{aligned}$$

For the continutiy, we simply modify the proof of Lemma 2.5 in [22]. Let u, $v \in W^{1/2,2}(\Gamma)$ and define

$$I(x,y) = \frac{|\beta(u(x)) - \beta(v(x)) - \beta(u(y)) + \beta(v(y))|^2}{|x-y|^n},$$

so that $|\beta(u) - \beta(v)|^2_{W^{1/2,2}(\Gamma)} = \int_\Gamma \int_\Gamma I(x,y)$. We have the two upper bounds

$$I(x,y) \le 2\,\mathrm{Lip}(\beta)^2 \frac{|u(x) - v(x)|^2 + |u(y) - v(y)|^2}{|x-y|^n}$$

and

$$\begin{aligned}
&I(x,y) \\
&\quad\le 2 \frac{|\beta(u(x)) - \beta(u(y))|^2 + |\beta(v(x)) - \beta(v(y))|^2}{|x-y|^n} \\
&\quad\le 2\,\mathrm{Lip}(\beta)^2 \frac{|u(x) - u(y)|^2 + |v(y) - v(x)|^2}{|x-y|^n} \\
&\quad= 2\,\mathrm{Lip}(\beta)^2 \frac{|u(x) - u(y)|^2 + |v(y) - v(x) + (u(x) - u(y)) - (u(x) - u(y))|^2}{|x-y|^n} \\
&\quad\le 2\,\mathrm{Lip}(\beta)^2 \frac{|u(x) - u(y)|^2 + 2|u(x) - v(x) - u(y) + v(y)|^2 + 2|u(x) - u(y))|^2}{|x-y|^n} \\
&\quad\le 6\,\mathrm{Lip}(\beta)^2 \frac{|u(x) - u(y)|^2 + |u(x) - v(x) - u(y) + v(y)|^2}{|x-y|^n}.
\end{aligned}$$

Now, for $\epsilon > 0$, define

$$A_\epsilon = \{(x,y) \in \Gamma \times \Gamma \mid |u(x) - v(x)|^2 + |u(y) - v(y)|^2 \ge \epsilon |x-y|^n\}.$$

We then have

$$
\begin{aligned}
|\beta(u)-\beta(v)|^2_{W^{1/2,2}(\Gamma)} &= \int_\Gamma\int_\Gamma I(x,y)\\
&= \int_\Gamma\int_\Gamma I(x,y)\chi_{\{\Gamma\times\Gamma\setminus A_\epsilon\}}(x,y) + \int_\Gamma\int_\Gamma I(x,y)\chi_{A_\epsilon}(x,y)\\
&\leq 2\operatorname{Lip}(\beta)^2\epsilon|\Gamma|^2 + 6\operatorname{Lip}(\beta)^2\int_\Gamma\int_\Gamma \frac{|u(x)-u(y)|^2}{|x-y|^n}\chi_{A_\epsilon}(x,y)\\
&\quad + 6\operatorname{Lip}(\beta)^2|u-v|^2_{W^{1/2,2}(\Gamma)}
\end{aligned}
$$

and as $u\to v$, the right hand side tends to

$$2\operatorname{Lip}(\beta)^2\epsilon|\Gamma|^2$$

since $|A_\epsilon|\to 0$ in the limit and using the DCT. Since ϵ is arbitrary, we get $|\beta(u)-\beta(v)|_{W^{1/2,2}(\Gamma)}\to 0$. □

Lemma 4.A.1 (Bounded linear extension)**.** Let $Z_d\subset Z$ be a continuous and dense embedding of Banach spaces. Let $A\colon Z_d\to Y$ be an operator between Banach spaces such that

$$\|Az\|_Y\leq C\,\|z\|_Z \quad \text{for } z\in Z_d.$$

Then A can be uniquely extended to a bounded linear map $\bar{A}\colon Z\to Y$ such that $\bar{A}|_{Z_d}=A$ and

$$\bar{A}z := \lim_{n\to\infty} Az_n \quad \text{in } Y\text{, for } z_n\in X_d \text{ with } z_n\to z \text{ in } X.$$

Furthermore, the operator norm is preserved.

4.A.1 Subdifferentials

The following is taken from [91, §3.2]. Below, X is a Banach space and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X^* and X.

Definition 4.A.2 (Generalised directional derivative)**.** If $\psi\colon U\subseteq X\to\mathbb{R}$ is locally Lipschitz, the *generalised directional derivative* (in the sense of Clarke) at $x\in U$ in the direction $v\in X$ is defined by

$$\psi^0(x;v)=\limsup_{y\to x,\lambda\to 0}\frac{\psi(y+\lambda v)-\psi(y)}{\lambda}.$$

This is always well-defined in contrast to the usual directional derivative.

Definition 4.A.3 (Classical directional derivative)**.** For all $\psi\colon U \subseteq X \to \mathbb{R}$, the *classical directional derivative* is

$$\psi'(x;v) := \lim_{\lambda\to 0} \frac{\psi(x+\lambda v)-\psi(x)}{\lambda}$$

if the limit exists.

Definition 4.A.4 (Clarke subdifferential or generalised gradient)**.** Let $\psi\colon U \subseteq X \to \mathbb{R}$ be locally Lipschitz. The *Clarke subdifferential* of ψ at $x \in U$ is defined by

$$\partial\psi(x) := \{f \in X^* \mid \psi^0(x;v) \geq \langle f, v\rangle \text{ for all } v \in X\} \subset X^*.$$

Definition 4.A.5 (Regular function)**.** A locally Lipschitz function $\psi\colon U \to \mathbb{R}$ is *regular* (in the sense of Clarke) at $x \in U$ if $\psi'(x,v)$ exists and $\psi'(x;v) = \psi^0(x;v)$ for all $v \in X$.

Lemma 4.A.6. For all the x and v such that $\psi'(x;v)$ exists, we have $\psi'(x;v) \leq \psi^0(x;v)$.

Definition 4.A.7 (Gâteaux derivative)**.** We say that ψ is *Gâteaux differentiable* if the classical directional derivative exists and there is an element $\psi'_G(x) \in X^*$ such that

$$\psi'(x;v) = \langle \psi'_G(x), v\rangle$$

for all $v \in X$.

Definition 4.A.8 (Convex subdifferentiable)**.** Let $U \subset X$ be open and convex and let $\psi\colon U \to \mathbb{R}$ be convex. An element $f \in X^*$ is the *convex subdifferential* of ψ at $x \in U$ if

$$\psi(v) \geq \psi(x) + \langle f, v-x\rangle \text{ for all } v \in U.$$

The set of all such subdifferentials at x is denoted $\partial_1\psi(x)$.

We will refer to the convex subdifferential as just the subdifferential.

Lemma 4.A.9. Let $\psi\colon U \subset X \to \mathbb{R}$ be convex where U is open and convex. Then

1. The Clarke subdifferential agrees with the convex subdifferential, i.e., $\partial\psi(x) = \partial_1\psi(x)$.
2. The function ψ is locally Lipschitz and regular on U.
   (a) This implies that if the Gâteaux derivative $\psi'_G(x)$ of ψ exists, then

$$\partial\psi(x) = \{\psi'_G(x)\}.$$

Conclusions

There are many interesting problems on evolving surfaces worth studying that we did not have time for; we discuss a few of them now.

Time-periodic problems If the hypersurface $\Gamma(t)$ evolves periodically, so that $\Gamma(0) = \Gamma(T)$, do there exist periodic solutions to parabolic equations? The simplest example would be the linear heat equation with the source term satisfying $\int_0^T \int_{\Gamma(t)} f = 0$ supplemented with a fixed initial mean value $\overline{u(0)} = c$ and the periodicity condition $u(0) = u(T)$. This problem has been solved with solutions in the class of Hölder continuous functions in [53] by Elliott and Fritz, but a theory for weak (Sobolev) solutions is still missing. Some progress was made (using the theory of Fredholm operators) by Alphonse, Elliott, and Fritz but a final step is unsolved.

Asymptotics Suppose that $\Gamma(t) \to \Gamma_\infty$ in some sense as $t \to \infty$. Does the solution u of the heat equation converge $u(t) \to u_\infty$ as $t \to \infty$? If so, does u_∞ solve the corresponding stationary problem on Γ_∞?

Blow-up of solutions Take an equation on a stationary domain that exhibits blow-up of solutions. If we consider the corresponding equation on a domain that is strictly increasing, does the associated solution still show the blow-up behaviour? How does the rate of growth of the domain relate to the blow-up (if present)?

Free boundary problems The problems we have considered in this thesis all involve evolving domains or surfaces where the evolution has been prescribed. Can we formulate and study a problem where the evolution is unknown (in the abstract setting if possible)? For example, the domain movement could be given by the solution of a PDE which itself could be coupled to another equation on the evolving domain. This could lead to a system of equations and fixed-point methods may be useful.

Other definitions of solutions One could think about different and weaker notions of solutions than the ones we have considered. Regularity in space and time of solutions is also something that we did not discuss in much detail.

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